Improved ε expansion for three-dimensional turbulence: Two-loop renormalization near two dimensions

L. Ts. Adzhemyan,[1] J. Honkonen,[2] M. V. Kompaniets,[1] and A. N. Vasil'ev[1]
[1]*Department of Theoretical Physics, St. Petersburg University, Uljanovskaja 1, St. Petersburg, Petrodvorets 198504, Russia*
[2]*Theoretical Physics Division, Department of Physical Sciences, University of Helsinki, P.O. Box 64, FI-00014, Finland*
(Received 29 July 2004; published 17 March 2005)

An improved ε expansion in the d-dimensional ($d>2$) stochastic theory of turbulence is constructed at two-loop order, which incorporates the effect of pole singularities at $d\to 2$ in coefficients of the ε expansion of universal quantities. For a proper account of the effect of these singularities, two different approaches to the renormalization of the powerlike correlation function of the random force are analyzed near two dimensions. By direct calculation, it is shown that the approach based on the mere renormalization of the nonlocal correlation function leads to contradictions at two-loop order. On the other hand, a two-loop calculation in the renormalization scheme with the addition to the force correlation function of a local term to be renormalized instead of the nonlocal one yields consistent results in accordance with the ultraviolet (UV) renormalization theory. The latter renormalization prescription is used for the two-loop renormalization-group analysis amended with partial resummation of the pole singularities near two dimensions, leading to a significant improvement of the agreement with experimental results for the Kolmogorov constant.

DOI: 10.1103/PhysRevE.71.036305 PACS number(s): 47.27.−i, 47.10.+g, 05.10.Cc

I. INTRODUCTION

The renormalization-group (RG) method in the theory of turbulence allows us to calculate various physical quantities—critical exponents and universal amplitude ratios—in the form of an expansion in a small parameter ε. The real value of this parameter is not small, however, therefore justified doubts arise as to whether this method is of any use for acceptable numerical estimates of the quantities studied. Until recently, practical calculations were carried out only in the simplest (one-loop) approximation and therefore it was not possible to assess how the next-to-leading terms of the expansion actually compare with the leading order at the real value of the parameter $\varepsilon=2$. In Refs. [1,2], this problem was analyzed in the example of calculation of the skewness factor and the Kolmogorov constant in the inertial range. The calculation showed that the relative part of the two-loop. correction is indeed large, of the order of 100% in the real space dimension $d=3$. This contribution, however, rapidly decreases with the growth of d: already for $d=5$ it yields only 30% and in the limit $d\to\infty$ it decreases to 10%. On the contrary, when the space dimension decreases from $d=3$ to $d=2$, a drastic growth of the correction term is observed.

Analysis of the dependence of the coefficients of the ε expansion on the space dimension has revealed that this property is connected with the divergence of some graphs in the limit $d\to 2$, and the singularities in $d-2\equiv 2\Delta$ accumulate with the order of the perturbation expansion. Contributions of these graphs turned out to give rise to the large value of the correction term also at $d=3$. Thus, satisfactory quantitative results may be expected only after summing, at least approximately, the contributions of the most singular graphs at all orders of the ε expansion. Such a summation has been carried out in Ref. [3] with the use of an additional renormalization and double expansion in ε and Δ and with the result of a significant relative reduction of the correction term and improvement of the agreement with experiment. The calculation in Ref. [3] has been carried out in the two-loop approximation for both the usual ε expansion and the double (ε,Δ) expansion. These expansions were used as complementary to each other to arrive at the final result—an approach distinguishing Ref. [3] from Refs. [4,5], in which the one-loop calculation in the (ε,Δ) expansion was carried out.

In Ref. [4], where the idea of the double expansion and subsequent extrapolation of the results from the starting point $d=2,\varepsilon=0$ to the physical point $d=3,\varepsilon=2$ was first applied to the stochastic Navier-Stokes problem, as well as later in Ref. [5], this expansion was used as an alternative to the usual ε expansion at $d>2$. The reason for the attention paid to this approach was that in this scheme—in contrast to the usual way of renormalization in the model with $d>2$, where only one quantity, the coefficient of viscosity, is renormalized—another quantity, the random force, must be renormalized as well. Therefore, it was definitely of interest to find out the consequences of this new physical factor. The results of Refs. [4,5] in this respect, however, are drastically different due to different approaches to the renormalization of the random force.

Renormalization of the random force was introduced in stochastic hydrodynamics already in Ref. [6], devoted to an analysis of the kinetics of near-equilibrium hydrodynamic fluctuations, but there it was of the usual multiplicative character. In Ref. [4], an analogous renormalization was applied to the stochastic theory of turbulence with an unexpected result: violation of the Kolmogorov hypothesis of the independence of correlation functions of viscosity in the inertial range was predicted. In Ref. [5], the renormalization scheme used in Ref. [4] was criticized and a more complex scheme put forward with results corroborating the Kolmogorov hypothesis. Thus, the problem of the correct choice of the renormalization scheme turned out to be crucial in the use of

the double expansion. In the body of the present paper, a detailed analysis of this issue shall be given, while here we limit ourselves to a brief commentary.

Although renormalization of a model is usually carried out in perturbation theory order by order, it is always implied that certain relations hold to all orders. Violation of such relations in an incorrect renormalization appears in different ways in different renormalization schemes, but it is always a sign of inherent inconsistency of the scheme rendering it useless for applications. One of the main conditions of consistency of UV renormalization is that counterterms used to remove the divergences are local in space and time: they have to be polynomials in fields and their derivatives, or—in the Fourier representation—polynomials in frequencies and wave vectors. As was pointed out in Ref. [5], it was just this condition that was violated in Ref. [4] when a multiplicative renormalization of the random force was carried out, in analogy with Ref. [6]. The point is that in Ref. [6] the correlation function of the random force was proportional to an integer power of k^2 (k^2 in model A and k^0 in model B) so that the renormalization corresponding to stretching of it was tantamount to introduction of local counterterms. In Ref. [4], on the contrary, this correlation function is proportional to a noninteger power of k^2 and it is thus nonlocal, which renders multiplicative renormalization inconsistent. The solution of the problem put forward in Ref. [5] is, in fact, nonmultiplicative renormalization of the random force: the introduction of the counterterms necessary to remove divergences does not correspond to stretching of the original nonlocal correlation function of the random force but to adding a local term to it.

In the one-loop approximation, to which the authors of Refs. [4,5] restricted themselves, it is possible to remove the divergences of the graphs both in the nonlocal renormalization scheme of Ref. [4] and in the local scheme of Ref. [5], so that at this level both approaches seem equally acceptable. The two-loop calculation carried out in the present paper yields a direct confirmation that only the local scheme of Ref. [5] is consistent—in accord with the general statements of the renormalization theory. In the present paper, we pay considerable attention to this issue, because the nonlocal scheme has been repeatedly applied in fairly recent literature [7,8]. A technical account of the method, which allowed us to obtain the two-loop results announced in [3], is given here as well.

This paper is organized as follows. In Sec. II, we recall basic features of the (field-theoretic) renormalization procedure and the subsequent asymptotic analysis in the two-loop approximation well above the problematic two dimensions. Section III is devoted to a detailed argument showing why the multiplicative nonlocal renormalization fails at the two-loop order of the double expansion. In Sec. IV, the consistency of the local two-charge renormalization scheme is demonstrated by the results of the two-loop calculation in space dimensions $d \leqslant 2$ in which the technically simplest combined scheme of analytic and dimensional renormalization is unconditionally valid. Renormalization-group equations are set up in Sec. V with the subsequent two-loop solution for asymptotic analysis in the inertial range. Details of the method of calculation of universal quantities in the improved ε expansion are exposed in Sec. VI. Section VII contains discussion of the results and concluding remarks. Appendix A contains an overview of the two-loop RG analysis in a scheme in which renormalization is fixed—instead of extracting only UV-divergent contributions from perturbation expansion—by normalization conditions imposed on correlation and response functions. Finally, the fairly technical issue of the possibility of analytic continuation of the results obtained for $d \leqslant 2$ to space dimensions above 2 is discussed in Appendix B.

II. RENORMALIZATION OF THE MODEL IN A FIXED SPACE DIMENSION $d > 2$

The statistical model of the developed homogeneous isotropic turbulence of incompressible fluid is based on the stochastic Navier-Stokes equation

$$\nabla_t \varphi_i = \nu_0 \partial^2 \varphi_i - \partial_i \mathcal{P} + f_i, \quad \nabla_t \equiv \partial_t + (\varphi \partial). \tag{1}$$

Here, $\varphi_i(t,\mathbf{x})$ is the divergenceless velocity field, $\mathcal{P}(t,\mathbf{x})$ and $f_i(t,\mathbf{x})$ are the pressure and the transverse random force per unit mass, respectively, and ν_0 is the kinematic viscosity. For the random force f, a Gaussian distribution is assumed with zero mean and the correlation function

$$\langle f_i(t,\mathbf{x}) f_j(t',\mathbf{x}') \rangle \equiv D_{ij}(t,\mathbf{x};t',\mathbf{x}') = \frac{\delta(t-t')}{(2\pi)^d} \int d\mathbf{k}\, P_{ij}(\mathbf{k}) d_f(k) \exp[\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')], \tag{2}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projection operator and d is the dimension of the coordinate space. For the function $d_f(k)$, the following powerlike form is adopted in the RG approach:

$$d_f(k) = D_0 k^{4-d-2\varepsilon}. \tag{3}$$

The quantity $\varepsilon > 0$ in Eq. (3) plays the role of a formal expansion parameter. The value corresponding to the physical model is $\varepsilon = 2$, because for $\varepsilon \to 2$, $D_0 \sim (2-\varepsilon)$, we arrive at $d_f(k) \sim \delta(\mathbf{k})$, which corresponds to energy injection by infinitely large eddies.

The stochastic problem (1) and (2) is equivalent to a quantum-field-theoretic model with a doubled set of transverse vector fields $\Phi \equiv \{\varphi, \varphi'\}$ and the action [9,10]

$$S(\Phi) = \varphi' D \varphi'/2 + \varphi' [-\partial_t \varphi + \nu_0 \partial^2 \varphi - (\varphi \partial)\varphi], \tag{4}$$

where D is the correlation function of the random force (2), and the necessary integrals over $(t,\mathbf{x})$ and sums over vector indices are implied. Action (4) gives rise to the standard diagrammatic technique with the bare propagators whose $(t,\mathbf{k})$ representation is of the form

$$\langle \varphi(t) \varphi'(t') \rangle_0 = \theta(t-t') \exp[-\nu_0 k^2 (t-t')],$$

$$\langle \varphi' \varphi' \rangle_0 = 0, \tag{5}$$

$$\langle\varphi\varphi\rangle_0 = \frac{d_f(k)}{2\nu_0 k^2}\exp[-\nu_0 k^2|t-t'|],$$

where the common factor $P_{ij}(\mathbf{k})$ has been omitted for simplicity. The interaction in Eq. (4) brings about the three-point vertex $-\varphi'(\varphi\partial)\varphi=\varphi_i' V_{ijs}\varphi_j\varphi_s/2$ with the vertex factor $V_{ijs}=i(k_j\delta_{is}+k_s\delta_{ij})$, where $\mathbf{k}$ is the wave vector of the field φ'. The expansion parameter of the perturbation theory is the coupling constant $g_0\equiv D_0/\nu_0^3$.

Model (4) is logarithmic (i.e., the coupling constant g_0 is dimensionless) at $\varepsilon=0$. In the analytic renormalization scheme adopted here, the UV divergences have the form of the poles in ε in the correlation functions of the field $\Phi\equiv\{\varphi,\varphi'\}$. Dimensional analysis (power counting) shows that for $d>2$, superficial UV divergences can be present only in the one-particle-irreducible (1PI) functions $\Gamma_{\varphi'\varphi}$ and $\Gamma_{\varphi'\varphi\varphi}$. These divergences may be removed by counterterms of the form

$$\varphi'\partial^2\varphi, \quad \varphi'\partial_t\varphi, \quad \varphi'(\varphi\partial)\varphi \tag{6}$$

in the action. Due to symmetry reasons, however, in model (4) only one counterterm of all allowed by the dimensional analysis is actually generated. First, the spatial derivative acting on the field φ in the interaction term of action (4) can be transferred to the field φ' with the use of integration by parts. This means that the counterterms to the 1PI functions must contain at least one spatial derivative, so that the structure $\varphi'\partial_t\varphi$ cannot possibly appear. Second, from the Galilean symmetry of action (4) it follows that the last two structures of Eq. (6) can be brought about as counterterms only in the invariant combination $\varphi'\nabla_t\varphi$ with the Lagrangian derivative $\nabla_t=\partial_t+(\varphi\partial)$ from Eq. (1). This excludes also the structure $\varphi'(\varphi\partial)\varphi$. Thus, in the generic case we are left with a single counterterm of the form $\varphi'\partial^2\varphi$. In the special case $d=2$, however, a new UV divergence appears in the 1PI function $\Gamma_{\varphi'\varphi'}$.

Consider the renormalization of model (4) in the two-loop approximation in $d>2$. In this case, the only counterterm required is $\varphi'\partial^2\varphi$, which is generated by multiplicative renormalization of the viscosity in the corresponding term of action (4). We shall use the scheme of minimal subtractions (MS) in which the renormalization constants are determined by the relations

$$\nu_0=\nu Z_\nu, \quad D_0=g_0\nu_0^3=g\mu^{2\varepsilon}\nu^3,$$

$$g_0=g\mu^{2\varepsilon}Z_g, \quad Z_g=Z_\nu^{-3}. \tag{7}$$

Here, μ is the scale-setting parameter (the reference mass) in the MS scheme, ν is the renormalized viscosity, and g is the dimensionless renormalized charge. The only independent renormalization constant in Eq. (7) is that of the viscosity Z_ν. The amplitude of the correlation function of the random force D_0 is not renormalized, because no counterterm of the form $\varphi'\varphi'$ in action (4) is necessary. This leads to the relation between the renormalization constants of the charge and viscosity indicated in Eq. (7).

In the MS scheme, the renormalization constants are constructed as Laurent series in ε of the form "$1+\Sigma_{n\geqslant 1}a_n\varepsilon^{-n}$". In particular,

$$Z_\nu = 1+u\frac{a_{11}}{\varepsilon}+u^2\left(\frac{a_{22}}{\varepsilon^2}+\frac{a_{21}}{\varepsilon}\right)+\cdots=1+\sum_{n=1}^{\infty}u^n\sum_{k=1}^{n}a_{nk}\varepsilon^{-k}, \tag{8}$$

where

$$u\equiv\frac{g\bar{S}d}{32}, \quad \bar{S}_d\equiv\frac{S_d}{(2\pi)^d}, \quad S_d\equiv\frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{9}$$

and the coefficients a_{nk} depend only on d. Here S_d is the surface area of the unit sphere in d-dimensional space and Γ is Euler's Gamma function.

We shall determine the constant Z_ν from the requirement that the 1PI correlation function $\Gamma_{\varphi'\varphi}$ at zero frequency ($\omega=0$) is UV-finite, i.e., finite at $\varepsilon\to 0$ when expressed as a function of the renormalized variables ν and g determined by relations (7). With respect to vector indices, the function $\Gamma_{\varphi'\varphi}$ is proportional to the transverse projector $P_{ij}(\mathbf{p})$, where $\mathbf{p}$ is the external wave vector. In the following, we shall deal with the scalar coefficient of this projector obtained by the contraction of the indices i and j and division by $\mathrm{Tr}\,P=d-1$. In terms of the bare parameters ν_0 and $D_0=g_0\nu_0^3$, this scalar coefficient at $\omega=0$ assumes the form $-\nu_0 p^2$+sum of contributions of the n-loop graphs, each of which contains n pieces of $\langle\varphi\varphi\rangle_0$ lines (5) and, correspondingly, the factor D_0^n. Thus, in view of dimensional arguments,

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{d-1}=\nu_0 p^2\left[-1+\sum_{n=1}^{\infty}\left(\frac{D_0\bar{S}_d}{32\nu_0^3 p^{2\varepsilon}}\right)^n\gamma_{\varphi'\varphi}^{(n)}\right] \tag{10}$$

with dimensionless coefficients $\gamma_{\varphi'\varphi}^{(n)}$ which only depend on d and ε. The factors 32 and $\bar{S}_d$ in Eq. (10) have been introduced for convenience. To obtain the renormalized function $\Gamma_{\varphi'\varphi}$, the parameters D_0 and ν_0 in Eq. (10) have to be expressed in terms of ν, g, and μ according to definitions (7), which leaves the coefficients $\gamma_{\varphi'\varphi}^{(n)}$ intact. It is convenient to divide the result by νp^2 to arrive at the dimensionless quantity

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}=-Z_\nu+us^{2\varepsilon}\gamma_{\varphi'\varphi}^{(1)}Z_\nu^{-2}+(us^{2\varepsilon})^2\gamma_{\varphi'\varphi}^{(2)}Z_\nu^{-5}+\cdots \tag{11}$$

with u from Eq. (9) and $s\equiv\mu/p$.

The renormalization constant Z_ν is determined from the condition of cancellation of the poles in ε in relation (11). In the coefficient $\gamma_{\varphi'\varphi}^{(1)}$, there is a simple pole $\sim 1/\varepsilon$, whereas $\gamma_{\varphi'\varphi}^{(2)}$ contains poles $\sim 1/\varepsilon$ and $\sim 1/\varepsilon^2$, etc. For the two-loop calculation of Z_ν, the following contributions are needed:

$$\gamma_{\varphi'\varphi}^{(1)}=\frac{A}{\varepsilon}+B+\cdots, \tag{12}$$

$$\gamma^{(2)}_{\varphi'\varphi} = \frac{C}{\varepsilon^2} + \frac{D}{\varepsilon} + \cdots, \tag{13}$$

where the ellipsis stands for irrelevant corrections $O(\varepsilon)$ in $\gamma^{(1)}_{\varphi'\varphi}$ and $O(1)$ in $\gamma^{(2)}_{\varphi'\varphi}$.

Denoting the contribution of the order $u^n \sim g^n$ to the renormalization constant (8) by $Z^{(n)}_\nu$, from the condition of cancellation of the divergences (poles in ε) in Eq. (11), we infer

$$Z^{(1)}_\nu = \mathcal{L}_\varepsilon[us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi}], \tag{14}$$

$$Z^{(2)}_\nu = \mathcal{L}_\varepsilon[u^2 s^{4\varepsilon}\gamma^{(2)}_{\varphi'\varphi} - 2Z^{(1)}_\nu us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi}], \tag{15}$$

where $\mathcal{L}_\varepsilon$ stands for the operation of extraction of the UV-divergent part, which here consists of poles in ε.

When relation (12) is substituted in Eq. (14), the UV-finite term B does not contribute and the coefficient $s^{2\varepsilon}=1+2\varepsilon \log s+\cdots$ may be replaced by the unity. As a result, we obtain

$$Z^{(1)}_\nu = \frac{uA}{\varepsilon}. \tag{16}$$

Substituting this expression together with relations (12) and (13) in Eq. (15), we find

$$Z^{(2)}_\nu = \mathcal{L}_\varepsilon\left[u^2 s^{4\varepsilon}\left(\frac{C}{\varepsilon^2}+\frac{D}{\varepsilon}\right) - 2u^2 s^{2\varepsilon}\frac{A}{\varepsilon}\left(\frac{A}{\varepsilon}+B\right)\right]. \tag{17}$$

In the terms $\sim 1/\varepsilon$ we may replace $s^{n\varepsilon}\to 1$, whereas in contributions $\sim 1/\varepsilon^2$ also the second term in the expansion $s^{n\varepsilon}=1+n\varepsilon \log s+\cdots$ must be retained, which gives rise to a contribution of the form $\varepsilon^{-1}\log s=\varepsilon^{-1}\log(\mu/p)$ in $Z^{(2)}_\nu$. The presence of such a term in Z_ν is unacceptable, because renormalization constants must not contain any wave-number dependence by their very definition. The condition of vanishing of the term $\sim\varepsilon^{-1}\log s$ in Eq. (17) is

$$C = A^2 \tag{18}$$

for the coefficients of relations (12) and (13).

The recent two-loop calculation [1] confirms that relation (18) holds. Substituting it in Eq. (17), we obtain

$$Z^{(2)}_\nu = u^2\left[-\frac{A^2}{\varepsilon^2}+\frac{D-2AB}{\varepsilon}\right]. \tag{19}$$

The one-loop coefficient A in Eqs. (16), (17), and (19) has been known for quite a while,

$$A = -\frac{4(d-1)}{d+2}.$$

For the nontrivial next-to-leading coefficients D and B in Eq. (19), integral representations readily calculable for any given d have been obtained in Ref. [1].

That condition (18) holds thus imposing on Z_ν cancellation of the contributions $\sim\log s$ is not a coincidence, but a consequence of general principles of the theory of UV renormalization. The most important of them is the requirement that all counterterms must be local in space (i.e., polynomial in wave vectors). In model (4) this is so, because the counterterm giving rise to the renormalization of the parameter ν_0 has the form of νp^2 multiplied by a wave-number-independent coefficient, i.e., a polynomial function in $\mathbf{p}$. Therefore, in this model all consequences of the general conjectures of the theory of UV renormalization must hold, in particular independence of the renormalization constants of wave numbers to all orders in the perturbation theory as well as the critical scaling due to the RG equations with the ε-dependent critical dimensions of the velocity field φ and the frequency ω (more details are in Sec. V),

$$\Delta_\varphi = 1 - 2\varepsilon/3, \quad \Delta_\omega = 2 - 2\varepsilon/3. \tag{20}$$

These are exact relations without any corrections of higher order in ε. They are a consequence of connection (7) between the renormalization constants Z_g and Z_ν which, in turn, follows from the absence of renormalization of the nonlocal contribution with the correlation function of the random force in action (4). At the real value $\varepsilon=2$, quantities (20) assume the Kolmogorov values

$$\Delta_\varphi = -1/3, \quad \Delta_\omega = 2/3. \tag{21}$$

Condition (18) ensuring independence of the renormalization constant of the wave number in the MS scheme may appear in a different form in other renormalization schemes. We will illustrate this point in Appendix A in the example of the scheme with the "normalization point" (NP). The MS and NP schemes differ by a finite renormalization of the parameters g and ν, therefore all objective physical quantities, in particular critical dimensions (20), calculated in these schemes coincide.

Critical dimensions (20) do not depend on d and thus for them the problem of singularities in the limit $d\to 2$ mentioned in Sec. I is not relevant. There are, however, other important physical quantities such as the skewness factor, Kolmogorov constant, and critical dimensions of various composite operators to which this problem persists. It is important that for these quantities, the problem of anomalous scaling is absent, which cannot be treated in the framework of the model with massless injection (3) lacking a dimensional parameter to account for the external scale of turbulence.

For such quantities, contrary to Eq. (20), the solutions contain full series of the form

$$R(\varepsilon,d) = \sum_{k=0}^{\infty} R_k(d)\varepsilon^k, \tag{22}$$

and the coefficients $R_k(d)$ in the limit $d\to 2$ reveal singular behavior of the type $\sim(d-2)^{-k}\sim\Delta^{-k}(2\Delta\equiv d-2)$ giving rise to the growth of the relative part of the correction terms at $d\to 2$. The effect of these is fairly discernable also at the real value $d=3$, hence the natural desire to sum up contributions of the form $(\varepsilon/\Delta)^k$ at all orders of the ε expansion (22). This may be done with the aid of the double (ε,Δ) expansion [4,5]. The idea of such an "improved ε expansion" with the use of the local renormalization scheme [5] was explained in our Rapid Communication [3], where many important subtleties and details of calculations were, however, not re-

flected due to lack of space. In the present paper, we give a detailed exposition and start from the proof of inconsistency of the renormalization scheme proposed in Ref. [4].

III. CONSTRUCTION OF THE DOUBLE (ε,Δ) EXPANSION: PROOF OF THE INCONSISTENCY OF THE NONLOCAL RENORMALIZATION [4] IN THE TWO-LOOP APPROXIMATION

Model (4) is logarithmic (i.e., the bare coupling constant g_0 is dimensionless) at $\varepsilon=0$ in function (3) in arbitrary space dimension d. In a fixed dimension $d>2$, the value $\varepsilon=2$ corresponds to the "real problem." Calculations in the framework of the ε expansion have a rigorous meaning only in the vicinity of $\varepsilon=0$, whereas continuation of the results to the "real" value $\varepsilon=2$ is always understood as an extrapolation. In the scheme applicable for $d>2$ reviewed in Sec. II, this extrapolation corresponds to the continuation along the vertical ray from the point $(d,\varepsilon=0)$ to the point $(d,\varepsilon=2)$ in the (d,ε) plane. The same final point may be reached along a ray from any starting point $(d_0\neq d,\varepsilon=0)$ at which the model is logarithmic as well. The extrapolation along the ray starting from the origin $(d_0=2,\varepsilon=0)$ is, however, singled out, because at $d=2$ in model (4) an additional UV divergence (absent at $d>2$) occurs in the 1PI function $\Gamma_{\varphi'\varphi'}$. On such a ray we put

$$d=2+2\Delta, \quad \Delta/\varepsilon=\zeta=\text{const}. \tag{23}$$

The parameters ε and Δ are considered small of the same order and their ratio $\Delta/\varepsilon=\zeta$ a fixed constant [$\zeta=1/4$ in the extrapolation to the point $(d=3,\varepsilon=2)$].

Extraction of contributions of the order ε^m with $\Delta/\varepsilon=\text{const}$ corresponds to the account of all contributions of the form $\varepsilon^m(\varepsilon/\Delta)^n$ with any $n=0,1,2\ldots$ and $m+n=k$ in Eq. (22). Thus the use of the (ε,Δ) expansion in such a form is directly related to the problem of the account of the singularities at $\Delta\to 0$ pointed out in the discussion of relation (22).

It is worth emphasizing that the very process of extrapolation along a ray from the starting point $(d=2,\varepsilon=0)$ is inapplicable to description of two-dimensional turbulence in which the physics is totally different from the three-dimensional problem due to the appearance of the inverse energy cascade [11]. In Fig. 1, we have plotted the borderline curve BAC between the direct (normal) and inverse energy cascades obtained in Ref. [12]. The starting point of the extrapolation for the two-dimensional case $(d=2,\varepsilon=0)$ lies in the region of the direct cascade, whereas the final point $(d=2,\varepsilon=2)$ lies in the region of the inverse cascade. Thus the ray connecting these points intersects the borderline—the curve BAC—so that the extrapolation becomes impossible. However, the ray connecting the starting point $(d=2,\varepsilon=0)$ and a final point like $(d=3,\varepsilon=2)$ lies completely in the region of the direct cascade, therefore on such a ray the problem of the change of the cascade pattern does not arise. The rightmost point of the region of the inverse cascade (point A in Fig. 1) has the coordinate $d_A\simeq 2.06$ [12]. In the preceding discussion of the extrapolation along the vertical ray from the point $d,\varepsilon=0)$ to the point $(d,\varepsilon=2)$ at $d>2$, it should have been noted that the condition is not simply $d>2$, but $d>d_A=2.06$. From the practical point of view this is irrelevant, because we are interested in the space dimension $d=3$.

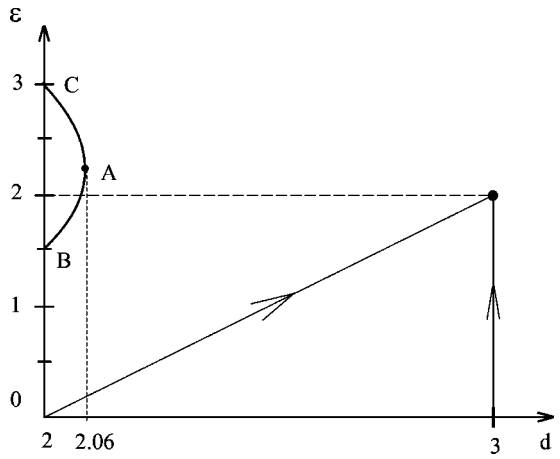

FIG. 1. The borderline BAC between the regions of parameter space d, ε corresponding to direct (to the right from the curve BAC) and inverse (to the left) energy cascades.

The idea of the double (ε,Δ) expansion together with the extrapolation along the ray $\Delta\sim\varepsilon$ of relation (23) in the context of the present problem was first put forward in Ref. [4]. The UV divergences are present not only in the 1PI function $\Gamma_{\varphi'\varphi}$ but also in $\Gamma_{\varphi'\varphi'}$ and appear in the form of poles in the parameters ε and Δ and linear combinations thereof, or, equivalently, as poles in ε with the fixed ratio $\Delta/\varepsilon\equiv\zeta=\text{const}$. To remove the additional divergences from the graphs of the 1PI function, $\Gamma_{\varphi'\varphi'}$ renormalization of the amplitude D_0 in the nonlocal correlation function of the random force (2) and (3) was used in Ref. [4], i.e., relations (7) between bare and renormalized parameters were replaced by

$$\nu_0=\nu Z_\nu, \quad D_0=g_0\nu_0^3=g\mu^{2\varepsilon}\nu^3 Z_D,$$

$$g_0=g\mu^{2\varepsilon}Z_g, \quad Z_g Z_\nu^3=Z_D \tag{24}$$

with a new renormalization constant Z_D which does not have an analog in Eq. (7).

It should be noted that the introduction of the additional constant Z_D breaks the last connection in Eq. (7) and its consequences (20). Therefore, the author of Ref. [4] has put forward the conjecture that in the scheme of the double (ε,Δ) expansion at the real value $\varepsilon=2$, the velocity field φ and the frequency ω have dimensions with values different from the Kolmogorov values (21). This is, of course, true, if renormalization relations (24) are used. We shall further show, however, that the renormalization scheme of Ref. [4] with relations (24) is not internally consistent. This is not obvious in the one-loop approximation, to which the author of Ref. [4] restricted himself, but becomes apparent already in the next two-loop approximation. In Ref. [5], another scheme of construction of the double (ε,Δ) expansion was put forward in which the last equality in Eq. (7) together with its consequences (20) and (21) are preserved. We shall deal with this approach in Sec. IV.

The main goal of this section is to prove that the scheme of multiplicative renormalization (24) contains intrinsic contradictions. To this end, consider representations similar to Eq. (11) for the 1PI functions $\Gamma_{\varphi'\varphi}$ and $\Gamma_{\varphi'\varphi'}$. According to Eq. (24), the amplitude D_0 in Eq. (10) now acquires the additional factor Z_D, therefore instead of relation (11) we now obtain

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)} = -Z_\nu + us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi}Z_\nu^{-2}Z_D + (us^{2\varepsilon})^2\gamma^{(2)}_{\varphi'\varphi}Z_\nu^{-5}Z_D^2 + \cdots. \tag{25}$$

The analogous relation for the 1PI function $\Gamma_{\varphi'\varphi'}$ is

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi'}|_{\omega=0}}{g\nu^3\mu^{2\varepsilon}p^{4-d-2\varepsilon}(d-1)} = Z_D + us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi'}Z_\nu^{-3}Z_D^2 + (us^{2\varepsilon})^2\gamma^{(2)}_{\varphi'\varphi'}Z_\nu^{-6}Z_D^3 + \cdots. \tag{26}$$

The expansion parameter is $u=g\overline{S_d}$ from Eq. (9). In Sec. II, the quantity d was considered a fixed parameter and therefore it was possible to treat $\overline{S_d}$ as a simple normalization factor. Here, d is determined by the relation (23) and in calculations within the usual MS scheme the quantity $\overline{S_d}$ should be expanded in the small parameter $\Delta\sim\varepsilon$. Following Ref. [4], we shall use the modified scheme $\overline{MS}$ (see, e.g., Ref. [13]), in which the quantity $\overline{S_d}$ is treated as a whole and not expanded in Δ. It is well known that the choice of scheme is not reflected in any physically significant results.

The constants Z are sought as series of form (8) and determined from the condition of cancellation of the UV divergences (poles in ε with $\Delta/\varepsilon=\mathrm{const}$) in relations (25) and (26). Denoting by $Z^{(n)}$ the contribution of order $u^n\sim g^n$ in any of these constants, we arrive at expressions similar to Eqs. (14) and (15): at the first order in $u\sim g$,

$$Z_\nu^{(1)} = \mathcal{L}_\varepsilon\{us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi}\},$$

$$Z_D^{(1)} = -\mathcal{L}_\varepsilon\{us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi'}\}, \tag{27}$$

and at the second order

$$Z_\nu^{(2)} = \mathcal{L}_\varepsilon\{u^2s^{4\varepsilon}\gamma^{(2)}_{\varphi'\varphi} + us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi}[Z_D^{(1)} - 2Z_\nu^{(1)}]\}, \tag{28}$$

$$Z_D^{(2)} = \mathcal{L}_\varepsilon\{-u^2s^{4\varepsilon}\gamma^{(2)}_{\varphi'\varphi'} + us^{2\varepsilon}\gamma^{(1)}_{\varphi'\varphi'}[3Z_\nu^{(1)} - 2Z_D^{(1)}]\}. \tag{29}$$

For calculation in the two-loop approximation, the following contributions are needed:

$$\gamma^{(1)}_{\varphi'\varphi} = \frac{A}{\varepsilon} + B + \cdots, \quad \gamma^{(1)}_{\varphi'\varphi'} = \frac{A'}{\varepsilon} + B' + \cdots, \tag{30}$$

$$\gamma^{(2)}_{\varphi'\varphi} = \frac{C}{\varepsilon^2} + \frac{D}{\varepsilon} + \cdots, \quad \gamma^{(2)}_{\varphi'\varphi'} = \frac{C'}{\varepsilon^2} + \frac{D'}{\varepsilon} + \cdots. \tag{31}$$

These are analogs of relations (12) and (13) with different coefficients, however, which now may depend on the ratio $\Delta/\varepsilon=\zeta$.

Substituting expressions (30) in Eq. (27), we find the one-loop contributions to the renormalization constants,

$$Z_\nu^{(1)} = \frac{uA}{\varepsilon}, \quad Z_D^{(1)} = -\frac{uA'}{\varepsilon}. \tag{32}$$

One-loop calculation yields the following values (first obtained in Ref. [4]):

$$A=-1, \quad A' = \frac{1}{2+\zeta}. \tag{33}$$

In the one-loop approximation, there are no problems with $\log s$ in the constants Z, so that the multiplicative renormalization (24) appears quite acceptable.

Consider now two-loop contributions (28) and (29). Taking into account the already known one-loop expressions (32), we obtain

$$Z_\nu^{(2)} = \mathcal{L}_\varepsilon\left\{u^2s^{4\varepsilon}\left(\frac{C}{\varepsilon^2}+\frac{D}{\varepsilon}\right) + us^{2\varepsilon}\left(\frac{A}{\varepsilon}+B\right)\left(-\frac{uA'}{\varepsilon}-\frac{2uA}{\varepsilon}\right)\right\}, \tag{34}$$

$$Z_D^{(2)} = \mathcal{L}_\varepsilon\left\{-u^2s^{4\varepsilon}\left(\frac{C'}{\varepsilon^2}+\frac{D'}{\varepsilon}\right) + us^{2\varepsilon}\left(\frac{A'}{\varepsilon}+B'\right)\times\left(\frac{3uA}{\varepsilon}+\frac{2uA'}{\varepsilon}\right)\right\}. \tag{35}$$

The condition of cancellation of the contributions $\sim\varepsilon^{-1}\log s$ in Eq. (34) is

$$4C + 2A(-A' - 2A) = 0, \tag{36}$$

and in Eq. (35) analogously

$$-4C' + 2A'(3A + 2A') = 0. \tag{37}$$

Our two-loop calculation of the coefficients C and C' yields

$$C = 1 - \frac{1}{2(2+\zeta)},$$

$$C' = \frac{2}{(2+\zeta)(3+\zeta)} - \frac{3}{(3+\zeta)}. \tag{38}$$

Substitution in relations (36) and (37) of the calculated quantities (33) and (38) readily shows that condition (36) is satisfied, whereas Eq. (37) is not. This means that in $Z_\nu^{(2)}$ there is no "bad" contribution $\sim\varepsilon^{-1}\log s=\varepsilon^{-1}\log(\mu/p)$, while in $Z_D^{(2)}$ there is such a term,

$$\frac{2(1+\zeta)(4+3\zeta)}{(2+\zeta)^2(3+\zeta)}\varepsilon^{-1}\log(\mu/p), \tag{39}$$

whose coefficient is the expression on the left-hand side of Eq. (37).

Thus within the renormalization scheme of Ref. [4] according to relations (24) a dependence on the external wave numbers through $\log s=\log(\mu/p)$ appears in the renormalization constants, which is completely inacceptable by the very definition of the renormalization constants. It is not difficult to understand the reason for this: in scheme (24) there is a

violation of a fundamental principle of the general theory of UV renormalization—the requirement that all counterterms must be local (polynomial functions of external wave vectors) [13]. The introduction of the coefficient Z_D at the term $\sim\varphi' D\varphi'$ in action (4) with the nonlocal injection function (3) is tantamount to introduction of a nonlocal counterterm with the structure $p^{4-d-2\varepsilon}$. This feature takes the scheme discussed beyond the framework of the standard theory of UV renormalization with such unpleasant consequences as the appearance of the (unacceptable) dependence on wave numbers in the renormalization constants. This general line of argument motivated the authors of Ref. [5] to change the scheme of (ε,Δ) renormalization to conform to the requirement of the polynomial in wave-vector form of all the counterterms (localness), although in the one-loop calculation of Ref. [4] the inconsistency of the scheme proposed there does not appear explicitly.

It might be suggested to change relation (29) to exclude the wave-number-dependent contribution (39) from $Z_D^{(2)}$. Equation (29) was obtained, however, from the requirement that in the two-loop approximation all UV divergences—poles in ε—were removed from the renormalized 1PI function $\Gamma_{\varphi'\varphi'}$, so that any change of the form of $Z_D^{(2)}$ from Eq. (29) would lead to the appearance of poles in ε in the renormalized function $\Gamma_{\varphi'\varphi'}$.

The persistent opponent might say, "Who cares, I am not interested in the two-loop approximation, I am completely happy with the one-loop accuracy, where there are no problems." Here, the objection would be that elimination of UV divergences (poles in ε) to all orders in perturbation theory is not a caprice but a compelling necessity. If such poles are left, then there is no guarantee that results obtained at the lowest order of perturbation theory do not acquire corrections of the same order from the higher-order terms not accounted for (in fact, there is conviction in the opposite), i.e., lowest-order calculations become completely unreliable. Therefore, in particular, the conclusion of Ref. [4] that relations (20) are violated in the (ε,Δ) scheme is not correct; in the consistent renormalization scheme, these relations continue to hold [5].

In conclusion, let us point out that the "bad" contribution (39) in $Z_D^{(2)}$ vanishes at $\zeta=-1$, i.e., at $\Delta=-\varepsilon$ in Eq. (23). Then $d=2+2\Delta=2-2\varepsilon$ and energy injection (3) becomes local: $d_f\sim p^{4-d-2\varepsilon}=p^2$ (such a model was considered in Ref. [6]). In this case, the multiplicative renormalization (24) conforms to the requirement of local counterterms and the corresponding constants Z do not contain any dependence on $\log s$ in accordance with the general theory.

IV. CONSTRUCTION OF THE (ε,Δ) EXPANSION IN THE TWO-CHARGE MODEL WITH LOCAL COUNTERTERMS: TWO-LOOP CALCULATION OF THE RENORMALIZATION CONSTANTS

In the preceding section it was shown that in the (ε,Δ) scheme (23) the multiplicative renormalization [4] of the amplitude D_0 in Eq. (3) is not acceptable. The reason is that the counterterm with structure (3) is nonlocal $\sim k^{4-d-2\varepsilon}=k^{2-2\Delta-2\varepsilon}$ on rays (23).

Guided by the general theory of the UV renormalization, the authors of Ref. [5] put forward another scheme, in which a local counterterm $\sim k^2$ instead of the nonlocal one $\sim k^{2-2\Delta-2\varepsilon}$ is used to absorb singularities from the graphs of the 1PI function $\Gamma_{\varphi'\varphi'}$. This corresponds to addition of the term $\sim\varphi'\partial^2\varphi'$ to the action functional. In functional (4) with the correlation function D from Eqs. (2) and (3) there is no such term, so that upon the addition of the term $\sim\varphi'\partial^2\varphi'$ the renormalization ceases to be multiplicative. This would be inessential if our only goal was the elimination of divergences from Green's functions, which is quite possible by a nonmultiplicative renormalization. For the use of the standard technique of the RG, multiplicative renormalization is, however, necessary. This is why the authors of Ref. [5] proposed to consider a two-charge model in which to function (3) $\sim k^{4-d-2\varepsilon}=k^{2-2\Delta-2\varepsilon}$ the term $\sim k^2$ is added at the outset with an independent coefficient,

$$d_f(k)=D_{10}k^{2-2\Delta-2\varepsilon}+D_{20}k^2=g_{10}\nu_0^3k^{2-2\Delta-2\varepsilon}+g_{20}\nu_0^3k^2. \tag{40}$$

Here, the amplitude D_0 of Eq. (3) is denoted by D_{10}. The parameters g_{10} and g_{20} introduced in Eq. (40) play the role of two independent bare charges.

The contribution with D_{20} in relation (40) corresponds to thermal fluctuations. A model with this term only has been analyzed earlier in Ref. [6]. In the theory of turbulence, $D_{20}=0$ should be considered the "real value" of this parameter, since only the first term in Eq. (40) at $\varepsilon=2$ reproduces the pumping of energy by large-scale eddies. It will be shown below that vanishing of the bare parameter $g_{20}=D_{20}\nu_0^{-3}=0$ does not imply vanishing of the corresponding renormalized parameter g_2, so that in terms of renormalized parameters, function (40) gives rise to a two-charge model.

The unrenormalized action is, as before, functional (4), but now with the injection function (40) instead of Eq. (3) in the correlation function (2). In the adopted shorthand notation

$$S(\Phi)=\frac{1}{2}\varphi'(D_{10}k^{2-2\Delta-2\varepsilon}+D_{20}k^2)\varphi' + \varphi'[-\partial_t\varphi+\nu_0\partial^2\varphi-(\varphi\partial)\varphi]. \tag{41}$$

The propagators $\langle\varphi\varphi'\rangle_0$ and $\langle\varphi'\varphi'\rangle_0$ corresponding to action (41) maintain an earlier form (5), whereas $\langle\varphi\varphi\rangle_0$ is replaced by

$$\langle\varphi\varphi\rangle_0=\frac{(D_{10}k^{2-2\Delta-2\varepsilon}+D_{20}k^2)}{2\nu_0k^2}\exp[-\nu_0k^2|t-t'|]. \tag{42}$$

We are interested in the region $\varepsilon>0$ and $\Delta>0$ in Eq. (23). In this region, in model (41) the additional problem of "Λ divergences" arises which was absent in model (4) with injection function (3). Let us explain this in more detail. Wave-vector integrals—with the shorthand notation $\int dk\ldots$—corresponding to the 1PI graphs discussed always reduce to "nearly logarithmic" ones in the present set of models. Their deviation from logarithmicity appears in the form of factors of the type k^α with a small exponent $\alpha=2m\Delta-2n\varepsilon$, where n and m are non-negative integers. The

exponent α is the wave-number dimension of the wave-vector integrals obtained upon all time integrations and may be calculated by the following simple rule: each loop integral over wave vectors contributes a term 2Δ to α, the term with D_{10} in Eq. (42) yields the contribution $-2\varepsilon-2\Delta$, but the term with D_{20} does not affect α at all. Thus it may readily be seen that if only the nonlocal term with D_{10} is left in Eq. (40) [i.e., if we return to model (3)], then all the exponents α in the graphs of $\Gamma_{\varphi'\varphi}$ and $\Gamma_{\varphi'\varphi'}$ at $\varepsilon>0$ and $\Delta>0$ become negative. All the integrals in the limit $k\to\infty$ converge, they may be carried out over the whole wave-vector space, and the divergences appear as poles in ε,Δ and their linear combinations.

However, in the model with injection (40)—due to the presence of the second term with D_{20}—at $\Delta>0$ wave-vector integrals appear with $\alpha>0$. They diverge in the limit $k\to\infty$ and thus require an UV cutoff Λ. As examples we quote the values of α in the graphs of interest for us. In the one-loop graphs of $\Gamma_{\varphi'\varphi}$: $\alpha=-2\varepsilon,2\Delta$; in the two-loop graphs: $\alpha=-4\varepsilon$, $-2\varepsilon+2\Delta,4\Delta$; in the one-loop graphs of $\Gamma_{\varphi'\varphi'}$: $\alpha=-4\varepsilon-2\Delta$, $-2\varepsilon,2\Delta$; in the two-loop graphs: $\alpha=-6\varepsilon-2\Delta,-4\varepsilon,-2\varepsilon+2\Delta$, 4Δ.

Thus, in the two-charge model (40) at $\Delta>0$ ($\varepsilon>0$ is always implied), some integrals have the Λ divergence at large k. To remove these divergences, an additional procedure of Λ renormalization procedure is needed which we shall discuss in the Appendix. At the moment, the important point is that after the Λ renormalization, the limit $\Lambda\to\infty$ may be taken with the result that divergences appear only in the form of poles in ε,Δ and their linear combinations. The same poles may be found within the "formal scheme," where all integrals are understood as an analytic continuation on the parameters ε and Δ from the region, where there are no Λ divergences.

In our case, this is the region of $\varepsilon>0$ and small (compared with ε) negative $\Delta<0$ (i.e., $d<2$). In this section, we shall consider results obtained in the framework of this "formal scheme." There is no UV-cutoff parameter Λ in this scheme, but the divergences appear in the form of poles in ε with $\Delta/\varepsilon=\text{const}$. The goal of the renormalization is removal of these poles. In Appendix ???, it will be shown that the results obtained this way coincide with those obtained at $\Delta>0$ after the Λ renormalization and subsequent limit $\Lambda\to\infty$.

The relations of multiplicative renormalization in the formal scheme are

$$D_{10}=g_{10}\nu_0^3=g_1\mu^{2\varepsilon}\nu^3,\quad g_{10}=g_1\mu^{2\varepsilon}Z_{g_1},$$

$$D_{20}=g_{20}\nu_0^3=g_2\mu^{-2\Delta}\nu^3Z_{D_2},\quad g_{20}=g_2\mu^{-2\Delta}Z_{g_2},$$

$$\nu_0=\nu Z_\nu,\quad Z_{g_1}Z_\nu^3=1,\quad Z_{g_2}Z_\nu^3=Z_{D_2},\tag{43}$$

with two independent renormalization constants for the viscosity ν_0 and the amplitude D_{20}; the amplitude D_{10} of the nonlocal correlation function of the random force is not renormalized. The renormalization constants Z_ν and Z_{D_2} are found from the condition that the 1PI functions $\Gamma_{\varphi'\varphi}|_{\omega=0}$ and $\Gamma_{\varphi'\varphi'}|_{\omega=0}$ are UV-finite (i.e., with $\Delta/\varepsilon=\text{const}$ there are no poles in ε). The dimensionless expansion parameters of the perturbation theory for these quantities are

$$\alpha_1\equiv\frac{D_{10}\overline{S_d}}{32\nu_0^3p^{2\varepsilon}},\quad \alpha_2\equiv\frac{D_{20}\overline{S_d}}{32\nu_0^3p^{-2\Delta}}\tag{44}$$

with $\overline{S_d}$ from Eq. (9). Instead of relation (10), we now have

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{d-1}=\nu_0p^2\left[-1+\sum_{\substack{n_1\geqslant0,n_2\geqslant0,\\ n_1+n_2\geqslant1}}\alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\varphi'\varphi}^{(n_1,n_2)}\right],\tag{45}$$

and the analogous expression for $\Gamma_{\varphi'\varphi'}$,

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi'}|_{\omega=0}}{d-1}=D_{10}p^{2-2\Delta-2\varepsilon}+D_{20}p^2\left[1+\sum_{\substack{n_1\geqslant0,n_2\geqslant-1,\\ n_1+n_2\geqslant1}}\alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\varphi'\varphi}^{(n_1,n_2)}\right].\tag{46}$$

In terms of the renormalized variables, relations (45) and (46) yield for the reduced dimensionless functions the following representations:

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}=-Z_\nu+Z_\nu\sum_{\substack{n_1\geqslant0,n_2\geqslant0,\\ n_1+n_2\geqslant1}}\alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\varphi'\varphi}^{(n_1,n_2)},\tag{47}$$

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi'}|_{\omega=0}}{(d-1)g_2\nu^3\mu^{-2\Delta}p^2}=\frac{u_1}{u_2}s^{2\varepsilon+2\Delta}+Z_{D_2}+Z_{D_2}\sum_{\substack{n_1\geqslant0,n_2\geqslant-1,\\ n_1+n_2\geqslant1}}\alpha_1^{n_1}\alpha_2^{n_2}\gamma_{\varphi'\varphi'}^{(n_1,n_2)},\tag{48}$$

where the expansion parameters α_1 and α_2 from Eq. (44) are expressed through the renormalized parameters according to relations (43),

$$\alpha_1=u_1s^{2\varepsilon}Z_\nu^{-3},\quad \alpha_2=u_2s^{-2\Delta}Z_{D_2}Z_\nu^{-3}.\tag{49}$$

Here, $u_1=g_1\overline{S_d}/32$, $u_2=g_2\overline{S_d}/32$, and $s\equiv\mu/p$. Dependence on ε of the coefficient functions $\gamma_{\varphi'\varphi}^{(n_1,n_2)}$ and $\gamma_{\varphi'\varphi'}^{(n_1,n_2)}$ in Eqs. (47) and (48) is determined by relations of the form of Eqs. (30) and (31), in which the ($\zeta=\Delta/\varepsilon$)-dependent coefficients A, B, C, D, A', B', C', and D' now acquire subscripts corresponding to the superscripts (n_1,n_2) of the quantities $\gamma^{(n_1,n_2)}$. In the one-loop approximation, the following analogs of relations (30) are needed:

$$\gamma_{\varphi'\varphi}^{(i,k)}=\frac{A_{i,k}}{\varepsilon}+B_{i,k},\quad \gamma_{\varphi'\varphi'}^{(i,k)}=\frac{A'_{i,k}}{\varepsilon}+B'_{i,k},\tag{50}$$

with the index sets $(i,k)=\{(1,0),(0,1)\}$ for $\gamma_{\varphi'\varphi}$ and the sets $(i,k)=\{(2,-1),(1,0),(0,1)\}$ for $\gamma_{\varphi'\varphi'}$. In the two-loop ap-

proximation, the following analogs of relations (31) have to be included:

$$\gamma_{\varphi'\varphi}^{(i,k)} = \frac{C_{i,k}}{\varepsilon^2} + \frac{D_{i,k}}{\varepsilon}, \quad \gamma_{\varphi'\varphi'}^{(i,k)} = \frac{C'_{i,k,}}{\varepsilon^2} + \frac{D'_{i,k}}{\varepsilon}, \tag{51}$$

with the sets $(i,k)=\{(2,0),(1,1),(0,2)\}$ for $\gamma_{\varphi'\varphi}$ and $(i,k)=\{(3,-1),(2,0),(1,1),(0,2)\}$ for $\gamma_{\varphi'\varphi'}$. For calculation of the constants Z in the one-loop approximation, which was carried out in Ref. [5], only constants A and A' from Eq. (50) are needed. For our two-loop calculation, all constants in Eqs. (50) and (51) are necessary.

The constants Z_ν and Z_{D_2} are determined from the condition of cancellation of all UV divergences (poles in ε with $\Delta/\varepsilon=\text{const}$) in Eqs. (47) and (48). Denoting by $Z^{(n)}$ the contribution of order $u^n \sim g^n$ with respect to the set of charges u_1 and u_2 to any constant, we obtain at the first order in $u \sim g$

$$Z_\nu^{(1)} = \mathcal{L}_\varepsilon[u_1 s^{2\varepsilon}\gamma_{\varphi'\varphi}^{(1,0)} + u_2 s^{-2\Delta}\gamma_{\varphi'\varphi}^{(0,1)}],$$

$$Z_{D_2}^{(1)} = -\mathcal{L}_\varepsilon\left[\frac{u_1^2}{u_2}s^{4\varepsilon+2\Delta}\gamma_{\varphi'\varphi'}^{(2,-1)} + u_1 s^{2\varepsilon}\gamma_{\varphi'\varphi'}^{(1,0)} + u_2 s^{-2\Delta}\gamma_{\varphi'\varphi'}^{(0,1)}\right], \tag{52}$$

and at the second order

$$Z_\nu^{(2)} = \mathcal{L}_\varepsilon\{u_1^2 s^{4\varepsilon}\gamma_{\varphi'\varphi}^{(2,0)} + u_1 u_2 s^{2\varepsilon-2\Delta}\gamma_{\varphi'\varphi}^{(1,1)} + u_2^2 s^{-4\Delta}\gamma_{\varphi'\varphi}^{(0,2)} + u_1 s^{2\varepsilon}\gamma_{\varphi'\varphi}^{(1,0)}[-2Z_\nu^{(1)}] + u_2 s^{-2\Delta}\gamma_{\varphi'\varphi}^{(0,1)}[Z_{D_2}^{(1)} - 2Z_\nu^{(1)}]\}, \tag{53}$$

$$Z_{D_2}^{(2)} = -\mathcal{L}_\varepsilon\left\{\frac{u_1^3}{u_2}s^{6\varepsilon+2\Delta}\gamma_{\varphi'\varphi'}^{(3,-1)} + u_1^2 s^{4\varepsilon}\gamma_{\varphi'\varphi'}^{(2,0)} + u_1 u_2 s^{2\varepsilon-2\Delta}\gamma_{\varphi'\varphi'}^{(1,1)} + u_2^2 s^{-4\Delta}\gamma_{\varphi'\varphi'}^{(0,2)} + \frac{u_1^2}{u_2}s^{4\varepsilon+2\Delta}\gamma_{\varphi'\varphi'}^{(2,-1)}[-3Z_\nu^{(1)}]\right.$$
$$\left. + u_1 s^{2\varepsilon}\gamma_{\varphi'\varphi'}^{(1,0)}[Z_{D_2}^{(1)} - 3Z_\nu^{(1)}] + u_2 s^{-2\Delta}\gamma_{\varphi'\varphi'}^{(0,1)}[2Z_{D_2}^{(1)} - 3Z_\nu^{(1)}]\right\}. \tag{54}$$

Substituting expressions (50) and (52), we find the one-loop contributions to the renormalization constants,

$$Z_\nu^{(1)} = \frac{1}{\varepsilon}(u_1 A_{1,0} + u_2 A_{0,1}),$$

$$Z_{D_2}^{(1)} = -\frac{1}{\varepsilon}\left(\frac{u_1^2}{u_2}A'_{2,-1} + u_1 A'_{1,0} + u_2 A'_{0,1}\right). \tag{55}$$

The coefficients A and A' here have been calculated in Ref. [5]. In our notation,

$$A_{1,0} = -1, \quad A_{0,1} = \frac{1}{\zeta}, \quad A'_{2,-1} = \frac{1}{2+\zeta},$$

$$A'_{1,0} = 2, \quad A'_{0,1} = \frac{1}{\zeta}. \tag{56}$$

In the present work, we have carried out the two-loop calculation and determined the coefficients B and B' in Eq. (50) together with C, C', D, and D' in Eq. (51). Let us quote the coefficients C and C' necessary at the moment,

$$C_{2,0} = 1 - \frac{1}{2(2+\zeta)},$$

$$C_{1,1} = -\frac{2}{\zeta(1-\zeta)}, \quad C_{0,2} = \frac{1}{2\zeta^2},$$

$$C'_{3,-1} = \frac{2}{(2+\zeta)(3+\zeta)} - \frac{3}{3+\zeta}, \quad C'_{0,2} = -\frac{1}{2\zeta^2},$$

$$C'_{2,0} = -1 + \frac{1}{2+\zeta} + \frac{3}{2\zeta},$$

$$C'_{1,1} = \frac{4}{\zeta(1-\zeta)} + \frac{1}{1-\zeta}. \tag{57}$$

To check the cancellation of the "bad" terms $\sim\varepsilon^{-1}\log s$ in Eqs. (53) and (54), only terms $\sim 1/\varepsilon^2$ from them are needed. They are determined in Eq. (51) by the coefficients C and C' from Eq. (57) and in the contributions with A and A' from Eqs. (50), (55), and (56). Substitution shows that all contributions with $\varepsilon^{-1}\log s$ in Eqs. (53) and (54) cancel as required.

A specific feature of the renormalization constant Z_{D_2} is that it contains terms $\sim 1/u_2$ [see Eq. (55)]. When such a Z_{D_2} is substituted in renormalization relations (43) in the expression for D_{20}, terms independent of u_2 appear (generation terms),

$$D_{20}\frac{\overline{S_d}}{32} = u_2\mu^{-2\Delta}\nu^3 Z_{D_2}$$
$$= \nu^3\mu^{-2\Delta}\left[u_2 - \frac{1}{\varepsilon}(u_1^2 A'_{2,-1} + u_1 u_2 A'_{1,0} + u_2^2 A'_{0,1}) + \cdots\right]. \tag{58}$$

Due to such terms, the condition $D_{20}=0$ does not lead to the trivial conclusion $u_2=0$, i.e., a nonvanishing value of the renormalized charge corresponds even to the zero (real) value of the bare charge.

The ellipsis in Eq. (58) stands for contributions of the two-loop order and higher, which contain terms $\sim u^n/\varepsilon^{n-1}$ with $n \geqslant 3$. In the region $u \sim \varepsilon$ (where the fixed point u_* of the RG lies, see Sec. V), they are of the same order in ε as the explicitly quoted one-loop contribution in Eq. (58). Therefore, to determine the connection between the charges u_1 and u_2 imposed by the condition $D_{20}=0$ (i.e., $Z_{D_2}=0$), the two-loop calculation of the constants Z is not sufficient. This is unimportant, however, in the following, because in the RG analysis of Sec. V the charges u_1 and u_2 are considered independent parameters.

We shall not quote here the fairly cumbersome expressions obtained by us for the constants B, B', D, and D' in Eqs. (50) and (51). Instead, we quote the two-loop expressions for the renormalization constants Z_ν and Z_{D_2} obtained with the use of them and relations (53)–(57) in the $\overline{MS}$ scheme (a detailed account of the method of calculation can be found in Ref. [1]),

$$Z_\nu = 1 - \frac{u_1}{\epsilon} + \frac{u_2}{\zeta\epsilon} - \frac{1}{2}\left[\frac{4\zeta+3}{(2+\zeta)\epsilon} + \frac{2\zeta+1}{\zeta\epsilon^2}\right]u_1^2 - \left[\frac{5\zeta+3}{\epsilon(1-\zeta)} + \frac{2}{(1-\zeta)\epsilon^2}\right]u_1u_2 - \frac{1}{2}\left[\frac{1}{\zeta\epsilon} + \frac{1}{\zeta^2\epsilon^2}\right]u_2^2 + \frac{1}{\epsilon}\left[u_1^2 + 4\frac{u_1u_2}{(1-\zeta)} - \frac{u_2^2}{\zeta}\right]R, \tag{59}$$

$$Z_{D_2} = 1 - \frac{u_1^2}{\epsilon u_2(2+\zeta)} - \frac{2u_1}{\epsilon} + \frac{u_2}{\zeta\epsilon} + \left[\frac{\zeta(13+19\zeta)}{2(3+\zeta)(2+\zeta)\epsilon} + \frac{2\zeta+1}{(3+\zeta)(2+\zeta)\epsilon^2}\right]u_1^3u_2^{-1} - \frac{1}{2}\left[\frac{34\zeta+19+6\zeta^2}{(2+\zeta)\epsilon} + \frac{(\zeta+4)(2\zeta+1)}{\zeta(2+\zeta)\epsilon^2}\right]u_1^2$$
$$- \frac{1}{2}\left[\frac{13+31\zeta}{\epsilon(1-\zeta)} + \frac{2(4\zeta+1)}{(1-\zeta)\zeta\epsilon^2}\right]u_1u_2 - \frac{1}{2}\left(\frac{3}{\zeta\epsilon} + \frac{1}{\zeta^2\epsilon^2}\right)u_2^2 + \frac{1}{\epsilon}\left[2\frac{u_1^3}{u_2(3+\zeta)} + 3u_1^2 + 6\frac{u_1u_2}{(1-\zeta)} - \frac{u_2^2}{\zeta}\right](R-1), \tag{60}$$

where

$$R = -0.168.$$

This number has been obtained by a computer calculation of a relatively simple but cumbersome twofold integral, through which all the nontrivial two-loop contributions in Eqs. (59) and (60) are expressed.

V. RENORMALIZATION-GROUP REPRESENTATION

The use of renormalized parameters as such does not solve the main problem of a large expansion parameter growing with the Reynolds number. It is, however, a necessary step towards the use of the method of the renormalization group which allows us to solve the problem by effective resummation of the perturbation theory. We shall consider as an example the equal-time pair correlation function

$$\langle \varphi_i(t,\mathbf{x})\varphi_j(t,\mathbf{x}')\rangle \equiv G_{ij}(\mathbf{r}), \quad \mathbf{r} \equiv \mathbf{x}-\mathbf{x}', \tag{61}$$

which is the most interesting quantity for us in the following. The Fourier transform of this function may be written as

$$G_{ij}(\mathbf{p}) = P_{ij}(\mathbf{p})G(p), \tag{62}$$

where $P_{ij}(\mathbf{p})$ is the transverse projection operator and $p \equiv |\mathbf{p}|$. Dimensional arguments lead to the following representation of the scalar function $G(p)$ from Eq. (62):

$$G(p) = \nu^2 p^{-d+2} R(s,g_1,g_2), \quad s = \frac{\mu}{p}, \tag{63}$$

where R is a dimensionless function of dimensionless arguments. We want to calculate $G(p)$ in the inertial range of the wave number p. Since in the present model (3) the external scale of turbulence has been put equal to infinity, this corresponds to the region $s=\mu/p \gg 1$. The perturbation expansion of $G(p)$ contains powers of the parameter s whose exponents grow without limit, due to which it is ill-suited for finding the sought asymptotic behavior $s \to \infty$. We shall briefly revisit the solution of this problem within the method of RG.

Since the fields $\Phi=\{\varphi,\varphi'\}$ in the present problem are not renormalized, the renormalized functions W^R differ from the unrenormalized ones $W=\langle \Phi\cdots\Phi\rangle$ only by the choice of variables and the form of perturbation expansion (g_1 and g_2 instead of g_{10} and g_{20}), and we may write

$$W^R(g_1,g_2,\nu,\mu,\ldots) = W(g_{10},g_{20},\nu_0,\ldots).$$

Here, $e_0 \equiv \{\nu_0, g_{10}, g_{20}\}$ is the set of all bare parameters, whereas $e \equiv \{\nu, g_1, g_2\}$ are their renormalized analogs, and the ellipsis stands for the arguments not affected by renormalization such as the coordinates, times, etc. The unrenormalized functions W do not depend on μ, while the renormalized functions W^R do because of the introduction of μ in renormalization relations (43). The independence of μ of the functions W is expressed by the equation $\tilde{\mathcal{D}}_\mu W=0$. Here, and henceforth, $\tilde{\mathcal{D}}_\mu \equiv \mu\partial_\mu$ with fixed bare parameters e_0. The equation $\tilde{\mathcal{D}}_\mu W=0$ written in terms of the renormalized functions $W^R=W$ and their arguments e, μ is the basic RG equation

$$\tilde{\mathcal{D}}_\mu W^R(g,\nu,\mu,\ldots) = \mathcal{D}_{\mathrm{RG}} W^R(g,\nu,\mu,\ldots) = 0, \qquad (64)$$

where $\mathcal{D}_{\mathrm{RG}}$ stands for the operation $\tilde{\mathcal{D}}_\mu$ expressed in terms of the renormalized variables,

$$\mathcal{D}_{\mathrm{RG}} \equiv \mathcal{D}_\mu + \beta_1 \partial_{g_1} + \beta_2 \partial_{g_2} - \gamma_\nu \mathcal{D}_\nu, \qquad (65)$$

where $\mathcal{D}_x \equiv x\partial_x$ for any variable x. The RG coefficient functions (the anomalous dimensions γ and the β functions) in Eq. (65) are defined as

$$\gamma_a \equiv \tilde{\mathcal{D}}_\mu \ln Z_a, \quad a \equiv \{\nu, g_1, g_2, D_2\},$$

$$\beta_i \equiv \tilde{\mathcal{D}}_\mu g_i, \quad i = 1,2. \qquad (66)$$

The term with $\mathcal{D}_\nu$ in Eq. (64) is written with the account of renormalization relation (43) for ν and definition γ_ν (66). From Eq. (66) and renormalization relations (43), it follows that

$$\beta_1(g_1,g_2) = g_1[-2\varepsilon - \gamma_{g_1}(g_1,g_2)], \qquad (67)$$

$$\beta_2(g_1,g_2) = g_2[2\Delta - \gamma_{g_2}(g_1,g_2)],$$

$$\gamma_{g_1} = -3\gamma_\nu, \quad \gamma_{g_2} = \gamma_{D_2} - 3\gamma_\nu. \qquad (68)$$

We are interested in the infrared (IR) asymptotics of small wave vectors $\mathbf{p}$ and frequencies ω of the renormalized functions W^R or, equivalently, large relative distances and time differences in the $(t,\mathbf{x})$ representation [in static objects like Eqs. (61)–(63), dependence on t or ω is absent]. It is determined by the IR-stable fixed point g_*, at which $\beta(g_*)=0$ for all β functions. The fixed point g_* is IR-stable, if real parts of all eigenvalues of the matrix $\omega_{ij} \equiv \partial\beta_i/\partial g_j|_{g=g_*}$ are strictly positive (see, e.g., Refs. [15,16]). Below it will be shown that in our model (41) the system of two β functions (67) and (68) in the region of our interest $\varepsilon > 0$, $\Delta > 0$ has an IR-stable fixed point $g_* = \{g_{1*}, g_{2*}\}$ with $g_{1*} \neq 0$, $g_{2*} \neq 0$.

In its presence it follows from the RG equations (64) that (see, e.g.. Refs. [16,17]) the sought asymptotics $W^R|_{\mathrm{IR}}$ of the Green function W^R has the following property of "IR scaling" [in the $(t,\mathbf{x})$ representation]:

$$W^R|_{\mathrm{IR}}(\lambda^{-\Delta_\omega} t, \lambda^{-1}\mathbf{x}) = \lambda^{\Delta_W} W^R|_{\mathrm{IR}}(t,\mathbf{x}),$$

$$\Delta_W = \sum_\Phi \Delta_\Phi, \qquad (69)$$

where $\mathbf{x}$ is the set of all coordinate variables and t all times, whereas $\lambda > 0$ is an arbitrary stretching parameter. Summation in expression (69) for Δ_W goes over all fields $\Phi = \{\varphi, \varphi'\}$ entering the function W^R. In Eq. (69), only those arguments of the function W^R are explicitly shown which are stretched under a given scale transformation.

The quantities Δ_ω and Δ_Φ in Eq. (69) are critical dimensions of the frequency ω and the fields $\Phi = \{\varphi, \varphi'\}$. They are all unambiguously (see, e.g.. Refs. [16,17]) expressed through the quantity $\gamma_\nu^* \equiv \gamma_\nu(g_*)$—the value of the RG function $\gamma_\nu(g)$, defined in Eq. (66), at the fixed point,

$$\Delta_\varphi = 1 - \gamma_\nu^*, \quad \Delta_{\varphi'} = d - \Delta_\varphi,$$

$$\Delta_\omega = 2 - \gamma_\nu^*, \quad \gamma_\nu^* \equiv \gamma_\nu(g_*). \qquad (70)$$

At the fixed point with $g_{1*} \neq 0$ and $g_{2*} \neq 0$, the values $\gamma_a^* \equiv \gamma_a(g_*)$ of RG functions (66) are readily found from the definition the fixed point $\beta_1(g_*) = \beta_2(g_*) = 0$ together with relations (67) and (68): $\gamma_{g_1}^* = -2\varepsilon$, $\gamma_{g_2}^* = 2\Delta$, $\gamma_\nu^* = 2\varepsilon/3$, $\gamma_{D_2}^* = 2\Delta + 2\varepsilon$. Substitution of $\gamma_\nu^* = 2\varepsilon/3$ in Eq. (70) leads to formulas (20) and their corollaries (21) for $\varepsilon = 2$. Thus, in the two-charge model (41) with the local renormalization [5], the critical dimensions of the velocity field φ and frequency ω at the real value $\varepsilon = 2$ retain their Kolmogorov values contrary to the conjecture of the author of Ref. [4].

Consider again function (63). It is a particular case of the function W^R and satisfies the RG equation (64): $\mathcal{D}_{\mathrm{RG}} G = 0$. A representation of the solution of Eq. (64) for $G(p)$ convenient for the asymptotic analysis at $p \to 0$ may be obtained with the aid of invariant variables $\overline{e} = \overline{e}(s,e)$ corresponding to the complete set of renormalized parameters $e \equiv \{\nu, g_1, g_2\}$. They are defined as solutions of the RG equations $\mathcal{D}_{\mathrm{RG}} \overline{e} = 0$ with the operator $\mathcal{D}_{\mathrm{RG}}$ from Eq. (65) and the normalization conditions $\overline{e} = e$ at $s = 1$. In terms of the invariant variables, the solution of the RG equation (64) for $G(p)$ may be represented as

$$G(p) = \nu^2 p^{2-d} R(s, g_1, g_2) = \overline{\nu}^2 p^{2-d} R(1, \overline{g_1}, \overline{g_2}). \qquad (71)$$

The right-hand side of Eq. (71) depends on s through the invariant variables $\overline{e}(s,e)$ only, whose asymptotic behavior in the limit $s \to \infty$—determined by the IR-stable fixed point (see below)—is simple: the invariant charges $\overline{g_1}$ and $\overline{g_2}$ tend to fixed values $g_{1*} = O(\varepsilon)$ and $g_{2*} = O(\varepsilon)$, whereas the invariant viscosity has simple powerlike asymptotics. It may be conveniently determined by expressing the invariant variables $\overline{e} = (\overline{\nu}, \overline{g_1}, \overline{g_2})$ in terms of the bare variables $e_0 = (\nu_0, g_{10}, g_{20})$ and the wave number p. According to definition, the bare variables e_0 as well as the invariant variables $\overline{e}$ satisfy the equation $\mathcal{D}_{\mathrm{RG}} e_0 = \tilde{\mathcal{D}}_\mu e_0 = 0$. The connection between the two sets of parameters is determined by the relations

$$\nu_0 = \overline{\nu} Z_\nu(\overline{g}), \quad g_{10} = \overline{g_1} p^{2\varepsilon} Z_{g_1}(\overline{g}),$$

$$g_{20} = \overline{g_2} p^{-2\Delta} Z_{g_2}(\overline{g}), \qquad (72)$$

valid because both sides in each of them satisfy the RG equation, and because at $s \equiv \mu/p = 1$ they coincide with relations (43) owing to the normalization conditions. Using the connection between renormalization constants $Z_g Z_\nu^3 = 1$ indicated in Eq. (43) and excluding these constants from the first two relations in Eq. (72), we find $g_{10}\nu_0^3 = D_{10} = \overline{g_1} p^{2\varepsilon} \overline{\nu}^3$, and from here

$$\overline{\nu} = (D_{10} p^{-2\varepsilon}/\overline{g_1})^{1/3},$$

which for the sought asymptotics $s \to \infty$ with the account of $\overline{g_1} \to g_{1*}$ yields

$$\overline{\nu} \to \overline{\nu}_* = (D_{10}/g_{1*})^{1/3} p^{-2\varepsilon/3}, \quad s \to \infty. \qquad (73)$$

Substituting this result in Eq. (71), we obtain

$$G(p) \simeq (D_{10}/g_{1*})^{2/3} p^{2-d-4\varepsilon/3} R(1, g_*), \quad s \to \infty. \quad (74)$$

This relation will be used in Sec. VI.

Let us make a remark about relations (72). According to renormalization relations (43), condition $D_{20} \sim g_{20} = 0$ [see the text following Eq. (40)] imposes the constraint $Z_{D_2}(g) \sim Z_{g_2}(g) = 0$ on the renormalized charges $g = \{g_1, g_2\}$. From the last relation in Eq. (72) it follows that the invariant charges $\overline{g} = \overline{g}(s, g)$ for any value of the variable $s \equiv \mu/p$ lie on the same constraining surface $Z_{g_2} = 0$ as the initial data $\overline{g}|_{s=1} = g$. Therefore, the limit values $g_* = \lim_{s\to\infty} \overline{g}(s, g)$ lie on the same surface $Z_{g_2} = 0$, i.e., the condition $D_{20} \sim g_{20} = 0$ is compatible with the RG analysis.

All said above is valid for any subtraction scheme; only the explicit form of the RG functions γ_a in Eqs. (66) and (67) depends on the choice of the scheme. Here, we shall quote results of the two-loop calculation in the $\overline{MS}$ scheme (Sec. IV). A brief discussion of the modification of formulas in the NP scheme is deferred to Appendix A. As said before, no physically significant results depend on the choice of the scheme.

In the MS and $\overline{MS}$ schemes, all RG functions γ_a are independent of ε. In model (41), they depend only on charges and the parameter $\zeta = \Delta/\varepsilon$. The two-loop expressions for the constants Z_a in Eq. (66) are given by Eqs. (59) and (60). In calculation of the quantities $\gamma_a = \widetilde{\mathcal{D}}_\mu \ln Z_a$ from Eq. (66), the operation $\widetilde{\mathcal{D}}_\mu$ may be replaced by $\mathcal{D}_{\mathrm{RG}}$ from Eq. (64) and the contributions with $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ omitted, since the quantities Z_a do not depend on μ and ν. Such a calculation yields

$$\gamma_\nu = 2(u_1 + u_2) + \frac{2(4\zeta + 3)u_1^2}{2 + \zeta} + 2(5\zeta + 3)u_1 u_2 - 4R(u_1 + u_2)^2 + \cdots, \quad (75)$$

$$\gamma D_2 = \frac{2(u_1 + u_2)^2}{u_2} - \frac{\zeta(13 + 19\zeta)u_1^3}{(2 + \zeta)u_2} + \frac{2(34\zeta + 19 + 6\zeta^2)u_1^2}{2 + \zeta} - 6u_1^2 + (13 + 31\zeta)u_1 u_2 + \frac{4(1 - R)(u_1 + u_2)^3}{u_2} + \cdots. \quad (76)$$

Let us recall that $u_1 \sim g_1$ and $u_2 \sim g_2$ are charges with a more convenient normalization (49), while the ellipsis stands for corrections of order $O(u^3)$.

Substituting quantities (75) and (76) in Eq. (67), we obtain expressions for the β functions in the two-loop approximation. Then from the conditions $\beta_1(g_*) = \beta_2(g_*) = 0$, coordinates of the fixed points $g_* \sim u_*$ may be found. In the framework of the ε expansion, there are three fixed points [5]: (i) the trivial fixed point $u_{1*} = 0, u_{2*} = 0$; (ii) the "kinetic" fixed point $u_{1*} = 0, u_{2*} \neq 0$; and (iii) the "Kolmogorov" fixed point $u_{1*} \neq 0, u_{2*} \neq 0$. In the region $\varepsilon > 0$, $\Delta > 0$ of interest for us, only the Kolmogorov fixed point is IR-stable, for which in the one-loop approximation

$$u_{1*} + u_{2*} = \frac{\varepsilon}{3} + O(\varepsilon^2), \quad u_{2*} = \frac{\varepsilon}{9(1 + \zeta)} + O(\varepsilon^2). \quad (77)$$

From relations (75) and (76), two-loop contributions $\sim \varepsilon^2$ to Eq. (77) may be found. We do not quote them, because coordinates of a fixed point $u_* \sim g_*$ do not have direct physical meaning and do depend on the choice of the subtraction scheme. Objective quantities independent of the subtraction scheme are the eigenvalues of the matrix $\omega_{ij} = \partial\beta_i/\partial g_j|_{g=g_*}$. In our problem, the ω matrix is a 2×2 matrix, whose two eigenvalues $\omega_\pm$ in the two-loop approximation at the Kolmogorov fixed point are

$$\omega_\pm = \left(\zeta + \frac{4}{3} \pm \frac{\sqrt{9\zeta^2 - 12\zeta - 8}}{3}\right)\epsilon + \frac{2}{9}\left\{-3 - 2R - 3\zeta \pm \frac{[4(1 + 3\zeta)R - 6 - 12\zeta - 9\zeta^2]}{\sqrt{9\zeta^2 - 12\zeta - 8}}\right\}\epsilon^2. \quad (78)$$

We quote also for reference the relatively simple expressions for the trace and determinant of the ω matrix, through which the eigenvalues $\omega_\pm$ are unambiguously expressed,

$$\mathrm{Tr}\,\omega = \omega_+ + \omega_- = \frac{2}{3}(3\zeta + 4)\epsilon - \frac{4}{9}(3\zeta + 3 + 2R)\epsilon^2, \quad (79)$$

$$\det \omega = \omega_+ \omega_- = \frac{4}{3}(3\zeta + 2)\epsilon^2 - \frac{4}{9}(2R + 1)(3\zeta + 2)\epsilon^3. \quad (80)$$

The one-loop contributions $\sim \varepsilon$ in Eqs. (77)–(79) and $\sim \varepsilon^2$ in Eq. (80) were obtained earlier in Ref. [5]. In the one-loop approximation, this fixed point g_* is IR-stable in the sector $\varepsilon > 0$, $\zeta > -2/3$ in the (ε, Δ) plane. When $\varepsilon > 0$ and $\zeta < -2/3$, both eigenvalues (78) are real and have different signs [this may be seen most easily from the one-loop contribution in Eqs. (79) and (80)]. With growth of ζ upon intersection of the borderline $\zeta_0 = -2/3$, both eigenvalues become positive and then, upon reaching the next borderline $2(1 - \sqrt{3})/3 \simeq -0.488$, the argument of the root in Eq. (78) becomes negative, i.e., the fixed point becomes an IR-attractive focus with $\omega_\pm = a \pm ib$ with $a > 0$. It remains such until the next borderline $2(1 + \sqrt{3})/3 \simeq 1.821$ is reached, upon passing which the root argument in Eq. (78) becomes positive again and both eigenvalues $\omega_\pm$ real and positive. For our "physical" ray $\zeta = 1/4 (d = 3)$ the fixed point g_* is an IR-attractive focus.

What was said above refers to the one-loop approximation. The account of the two-loop corrections in Eqs. (78)–(80) leads to a deformation of the borderlines of the region of IR stability, but the "physical" segment of ray (23) with $\zeta = 1/4$, $0 < \varepsilon \leqslant 2$ still remains in this region.

VI. SKEWNESS FACTOR AND KOLMOGOROV CONSTANT

The exponent of the power of the wave number in Eq. (74) is determined exactly and does not have corrections in the form of higher powers of ε. At the physical value $\varepsilon = 2$,

this exponent assumes the Kolmogorov value. To find the Kolmogorov constant, the amplitude of this function has to be calculated, which, however, can be done only approximately, because the corresponding ε expansion does not terminate. In calculation of the amplitude, apart from technical difficulties at two-loop order, a principal problem arises as well. It is connected with the necessity to express the answer for $G(k)$ in terms of the energy injection rate $\overline{\mathcal{E}}$ instead of the parameter D_{10} of the forcing correlation (40). The connection between D_{10} and $\overline{\mathcal{E}}$ is determined by an exact relation expressing $\overline{\mathcal{E}}$ in terms of the function $d_f(k)$ in the correlation function (2),

$$\overline{\mathcal{E}} = \frac{(d-1)}{2(2\pi)^d} \int d\mathbf{k}\, d_f(k). \tag{81}$$

Substituting here function (40) with $D_{20}=0$ [see the text following Eq. (40)] and introducing the UV cutoff $k \leqslant \Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}$ (the inverse dissipation length), we obtain the following connection between the parameters $\overline{\mathcal{E}}$ and D_{10}:

$$D_{10} = \frac{4(2-\varepsilon)\Lambda^{2\varepsilon-4}}{\overline{S_d}(d-1)}\overline{\mathcal{E}}. \tag{82}$$

Idealized injection by infinitely large eddies corresponds to $d_f(k) \propto \delta(\mathbf{k})$. More precisely, according to Eq. (81),

$$d_f(k) = \frac{2(2\pi)^d \overline{\mathcal{E}}\delta(\mathbf{k})}{d-1}. \tag{83}$$

In view of the relation

$$\delta(\mathbf{k}) = \lim_{\varepsilon\to 2}(2\pi)^{-d}\int d\mathbf{x}(\Lambda x)^{2\varepsilon-4}\exp(i\mathbf{k}\cdot\mathbf{x}) = S_d^{-1}k^{-d}\lim_{\varepsilon\to 2}[(4-2\varepsilon)(k/\Lambda)^{4-2\varepsilon}],$$

the powerlike injection with $d_f = D_{10}k^{4-d-2\varepsilon}$ and the amplitude D_{10} from Eq. (82) in the limit $\varepsilon \to 2$ from the region $0 < \varepsilon < 2$ gives rise to the δ sequence (83).

Relation (82) reveals that at fixed $\overline{\mathcal{E}}$, the quantity D_{10} depends on ε and it is necessary to take this dependence into account in the construction of the ε expansion for the Kolmogorov constant. On the other hand, it shows that the quantity $R(1, g_*)$ from Eq. (74) must have a singularity of the form $(2-\varepsilon)^{-2/3}$ in the limit $\varepsilon \to 2$: only in this case will the Kolmogorov constant in the model with the injection $d_f = D_{10}k^{4-d-2\varepsilon}$ and the amplitude D_{10} from Eq. (82) have a finite value in the limit $\varepsilon \to 2$. The measurable experimental Kolmogorov constant C_K in terms of the model with such pumping corresponds to the limiting value $\varepsilon = 2$, and we want to define its generalization $C_K(\varepsilon)$ for the whole interval $0 \leqslant \varepsilon \leqslant 2$. Obviously, such a generalization cannot be done unambiguously, because it is not possible to define the unambiguous dependence of the parameter D_{10} in Eq. (82) on ε at a fixed value of $\overline{\mathcal{E}}$.

Let us explain this in more detail. When deriving relation (82), we assumed that integral (81) for the injection $d_f = D_{10}k^{4-d-2\varepsilon}$ has an upper cutoff equal to the inverse dissipative length $\Lambda = (\overline{\mathcal{E}}/\nu_0^3)^{1/4}$. Such a cutoff is natural, but at the same time only orders of magnitude may be discussed, of course not the exact values. Therefore, there is nothing to prevent replacing in Eq. (82) the cutoff parameter Λ by $a\Lambda$ with a coefficient a of the order of unity, which yields the extra factor $a^{2\varepsilon-4}$ on the right-hand side of Eq. (82). This factor tends to unity at $\varepsilon \to 2$, hence it does not affect the physical (real) value of the Kolmogorov constant $C_K(\varepsilon = 2)$, but it does affect coefficients of the hypothetical ε expansion of the function $C_K(\varepsilon)$. Generalizing these observations, it may be stated that the physical content of the theory is not changed, if to the right-hand side of Eq. (82) an extra factor $F(\varepsilon)$ with $F(2)=1$ is added. In Ref. [20] (see also [16,17,19]), relation (82) without the extra factor $F(\varepsilon)$ was regarded as the definition of the quantity D_{10}. Other approaches to the definition of the function $C_K(\varepsilon)$ and its ε expansion [21–27] may be reduced to the introduction of a particular function $F(\varepsilon)$ with $F(2)=1$ on the right-hand side of relation (82).

Thus, ε expansion of the Kolmogorov constant in the model with the powerlike injection is not defined unambiguously. However, physical quantities independent of the amplitude D_{10} (universal quantities) do have a well-defined ε expansion. The skewness factor

$$\mathcal{S} \equiv S_3/S_2^{3/2} \tag{84}$$

is an example of such a quantity. In Eq. (84), S_n are structure functions defined by the relations

$$S_n(r) \equiv \langle[\varphi_r(t,\mathbf{x}+\mathbf{r}) - \varphi_r(t,\mathbf{x})]^n\rangle, \quad \varphi_r \equiv \frac{(\varphi_i r_i)}{|\mathbf{r}|}. \tag{85}$$

According to Kolmogorov theory, the structure function $S_2(r)$ in the inertial range is of the form

$$S_2(r) = C_K\overline{\mathcal{E}}^{2/3}r^{2/3}, \tag{86}$$

where C_K is the Kolmogorov constant with a simple connection with the Kolmogorov constant of the energy spectrum [18]. Although there is strong experimental evidence that the Kolmogorov scaling $S_n(r) \sim r^{n/3}$ does not hold in the inertial range for the structure functions of order $n \geqslant 4$, for the second-order structure function $S_2(r)$ the experimental situation about anomalous scaling [i.e., deviation of the power of r from the Kolmogorov value $2/3$ in Eq. (86) in the inertial range] is still controversial and in any case this deviation is small [28]. Therefore, we shall use the Kolmogorov asymptotic expression (86) for the second-order structure function $S_2(r)$ in the following analysis.

The structure function $S_3(r)$ may be found exactly in the inertial range [18],

$$S_3(r) = -\frac{12}{d(d+2)}\overline{\mathcal{E}}r, \tag{87}$$

which allows us—with account of Eqs. (84) and (86)—to relate the Kolmogorov constant and the skewness factor

$$C_K = \left[-\frac{12}{d(d+2)\mathcal{S}} \right]^{2/3}. \tag{88}$$

Of the three quantities $S_2(r)$, $S_3(r)$, and $\mathcal{S}$, only $\mathcal{S}$ has a unique ε expansion. Thus, relation (88) (valid only for the physical value $\varepsilon=2$) might be used to determine C_K using the calculated value $\mathcal{S}(\varepsilon=2)$.

To find the RG representation of skewness factor (84), it is necessary to have RG representations of the functions $S_2(r)$ and $S_3(r)$. The function $S_2(r)$ is connected with the Fourier transform of the pair correlation function $G(k)$ by the relation

$$S_2(r) = 2\int \frac{d\mathbf{k}}{(2\pi)^d} G(k) \left[1 - \frac{(\mathbf{k}\cdot\mathbf{r})^2}{(kr)^2} \right] \{1 - \exp[i(\mathbf{k}\cdot\mathbf{r})]\}, \tag{89}$$

therefore its RG representation may be found on the basis of RG representation (74). An analogous RG representation in the inertial interval may be written for the function $S_3(r)$. It is more convenient, however, to use the following exact result, an analog of expression (87):

$$S_3(r) = -\frac{3(d-1)\Gamma(2-\varepsilon)(r/2)^{2\varepsilon-3} D_{10}}{(4\pi)^{d/2}\Gamma(d/2+\varepsilon)}. \tag{90}$$

This relation is a manifest demonstration that the amplitude of the structure function, expressed in terms of D_{10}, has a singularity at $\varepsilon\to 2$; in this case it is $\sim(2-\varepsilon)^{-1}$. On substitution of Eq. (82) in Eq. (90), this singularity cancels the corresponding zero on the right-hand side of Eq. (82), leading for $S_3(r)$ to an expression finite at $\varepsilon=2$ and coinciding with Eq. (87).

Relations (74), (89), and (90) might serve as the basis for construction of the ε expansion of the skewness factor (84). However, an additional difficulty arises on this way. The point is that the powerlike dependence $S_2(r)\sim r^{2-2\varepsilon/3}$, determined from Eqs. (74) and (89), is only valid when $\varepsilon>3/2$, because for $\varepsilon<3/2$, integral (89) diverges at $k\to\infty$ [this means that the main contribution to $S_2(r)$ in this case is given by the term $\langle\varphi_r^2(t,\mathbf{x})\rangle$ independent of r]. However, the derivative $r\partial_r S_2(r)$ is free from this flaw, because, according to Eq. (89),

$$r\partial_r S_2(r) = 2\int \frac{d\mathbf{k}}{(2\pi)^d} G(k) \left[1 - \frac{(\mathbf{k}\cdot\mathbf{r})^2}{(kr)^2} \right] (\mathbf{k}\cdot\mathbf{r})\sin(\mathbf{k}\cdot\mathbf{r}). \tag{91}$$

Integral (91) is convergent for all $0<\varepsilon<2$. On the other hand, at the physical value $\varepsilon=2$, the amplitudes in $S_2(r)$ and $r\partial_r S_2(r)$ differ by a trivial factor $2/3$, therefore in Refs. [1–3] for the construction of the ε expansion, the following analog of the skewness factor was used:

$$Q(\varepsilon) \equiv \frac{r\partial_r S_2(r)}{|S_3(r)|^{2/3}} = \frac{r\partial_r S_2(r)}{[-S_3(r)]^{2/3}}. \tag{92}$$

The Kolmogorov constant and the skewness factor are expressed through the value $Q(\varepsilon=2)$ according to Eqs. (84), (86), and (87) by the relations

$$C_K = \frac{3Q(2)}{2}\left[\frac{12}{d(d+2)}\right]^{2/3}, \quad \mathcal{S} = -\left[\frac{2}{3Q(2)}\right]^{3/2}. \tag{93}$$

Quantity (92) may be calculated both in the double (ε,Δ) expansion and in the usual ε expansion. In the former case, the corresponding expansion is obtained on the basis of relations (74), (90), and (91) in the form

$$Q(\varepsilon,\zeta) = \varepsilon^{1/3}\sum_{k=0}^{\infty} \Psi_k(\zeta)\varepsilon^k. \tag{94}$$

The usual ε expansion of the quantity Q for dimensions $d>2$ has been obtained in Ref. [1],

$$Q(\varepsilon,d) = \varepsilon^{1/3}\sum_{k=0}^{\infty} Q_k(d)\varepsilon^k. \tag{95}$$

The connection between expansions (94) and (95) is revealed by investigation of singularities of the coefficients $Q_k(d)$ in Eq. (95) at $d\to 2$. An analysis of these singularities shows that in the vicinity of $d-2=2\Delta=0$, these coefficients may be expressed in a Laurent expansion,

$$Q_k(d) = \sum_{l=0}^{\infty} q_{kl}\Delta^{l-k}. \tag{96}$$

Substitution of expression (96) in Eq. (95) leads to the representation

$$Q(\varepsilon,d) = \varepsilon^{1/3}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (\varepsilon/\Delta)^k q_{kl}\Delta^l. \tag{97}$$

Changing variables in Eq. (97) to ε and $\zeta=\Delta/\varepsilon$, we arrive at expansion (94), in which

$$\Psi_k(\zeta) = \sum_{l=0}^{\infty} q_{lk}\zeta^{k-l}. \tag{98}$$

Relations (96) and (98) show that the alternative ε expansions (94) and (95) sum different infinite subsequences of double sum (97). In Ref. [3], a procedure of improvement of the ε expansion was proposed with the use of the mutually complementary information about the quantity Q contained in the partial sums of expansions (94) and (95),

$$Q^{(n)}_{\varepsilon,\Delta} \equiv \varepsilon^{1/3}\sum_{k=0}^{n-1} \Psi_k(\zeta)\varepsilon^k, \quad Q^{(n)}_{\varepsilon} \equiv \varepsilon^{1/3}\sum_{k=0}^{n-1} Q_k(d)\varepsilon^k, \tag{99}$$

where $n\geqslant 1$ is the number of loops.

Terms in the double sum (97) taken into account in $Q^{(n)}_{\varepsilon,\Delta}$ and $Q^{(n)}_{\varepsilon}$ have been schematically plotted in Fig. 2 in the form of dashed horizontal and vertical stripes, respectively.

All terms in the dashed area will be taken into account in the effective quantity

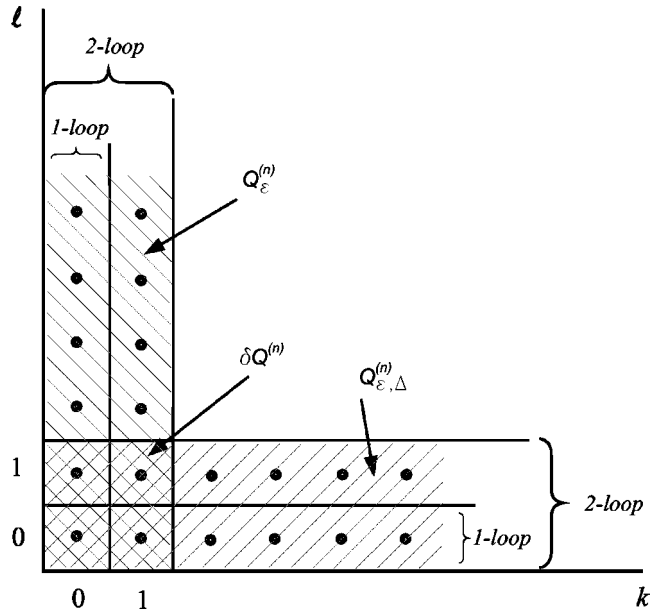

FIG. 2. Summations in the calculation of $Q_{\rm eff}^{(n)}$ in Eq. (100). Terms in the double sum (97) taken into account in $Q_{\varepsilon,\Delta}^{(n)}$ and $Q_\varepsilon^{(n)}$ correspond to the dashed horizontal and vertical stripes, respectively. The correction term $\delta Q^{(n)}$ corresponds to a sum over the double-dashed square.

$$Q_{\rm eff}^{(n)} = Q_\varepsilon^{(n)} + Q_{\varepsilon,\Delta}^{(n)} - \delta Q^{(n)}, \tag{100}$$

where

$$\delta Q^{(n)} \equiv \varepsilon^{1/3} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} (\varepsilon/\Delta)^k q_{kl} \Delta^l$$

is a subtraction term necessary to avoid double counting of terms with $k \leqslant n-1, l \leqslant n-1$ (the double-dashed square in Fig. 2). It may be found by taking the corresponding number of terms from expansions (96) or (98). From the point of view of the usual ε expansion (95), relation (100) may be interpreted as follows: in the $n-1$ first terms of the expansion, the coefficients $Q_k(d)$ from Eq. (95) are calculated exactly, but in all higher-order terms ($k \geqslant n$) they are calculated approximately with the account of $n-1$ first terms of their Laurent expansion (96).

Our two-loop calculation of the ε,Δ expansion of the quantity Q together with the two-loop calculation of Ref. [1] allowed us to obtain an improved ε expansion of the quantity Q at second order of perturbation theory [3]. For the Kolmogorov constant calculated according to Eq. (93) for $d=3$, it led to the result quoted in Table I.

In Table I, we have quoted for comparison the values of the Kolmogorov constant calculated according to Eq. (93) at first and second order of the usual ε expansion (C_ε), the double ε,Δ expansion ($C_{\varepsilon,\Delta}$), the contribution C_δ in Eq. (93) from the correction δQ^n in Eq. (100), and the value $C_{\rm eff}$ obtained from relations (93) and (100). In all the cases

TABLE I. One- and two-loop values of the Kolmogorov constant in the usual ε expansion (C_ε) and the double ε,Δ expansion ($C_{\varepsilon,\Delta}$); the contribution C_δ in Eq. (93) from the correction $\delta Q^{(n)}$ in Eq. (100); and the value $C_{\rm eff}$ from Eqs. (93) and (100).

n	C_ε	$C_{\varepsilon,\Delta}$	C_δ	$C_{\rm eff}$
1	1.47	1.68	1.37	1.79
2	3.02	3.57	4.22	2.37

quoted, the recommended experimental value of the Kolmogorov constant $C_{\rm exp}=2.01$ [29] lies between the values of the first and second approximation. However, the difference between these values is rather significant both in the ε expansion and in the (ε,Δ) expansion, let alone the leading terms of the ε expansion of the latter. For the improved ε expansion, i.e., for the quantity $C_{\rm eff}=C_\varepsilon+C_{\varepsilon,\Delta}-C_\delta$ calculated according to Eqs. (100) and (93), however, this difference is about three times smaller leading to far better agreement with the experimental data.

VII. CONCLUSION

In conclusion, we have presented a detailed comparison of two different renormalization schemes for the stochastic Navier-Stokes problem near two dimensions. By explicit two-loop calculation, we have shown that the nonlocal scheme of Ref. [4] cannot consistently be carried out beyond the leading one-loop approximation. On the contrary, our two-loop results confirm the consistency of the local renormalization scheme of Ref. [5] based on the general principles of the theory of UV renormalization.

The detailed explicit two-loop analysis of different renormalization schemes presented here is all the more important, because the inconsistent renormalization of nonlocal terms in dynamic models continues to appear in the literature [7,8].

The correct choice of the renormalization scheme is vital for a proper account of the effect on structure functions of the additional singularities appearing in the field-theoretic model in the limit $d \to 2$. Using the consistent local renormalization scheme, we have shown that a proper account of the "nearest singularity" in the coefficients of the ε expansion (95) leads to a significant improvement of the results of the two-loop RG calculation at $d=3$. We have analyzed the effect of this procedure at other d as well. It turned out to reduce significantly the relative contribution of the two-loop correction in the whole range considered $\infty > d \geqslant 2.5$. At the same time, this contribution remained large at $d=2$, which we think to be an effect of singularities at the next exceptional dimension $d=1$.

The proposed procedure of approximate summation of the ε expansion is, of course, applicable not only to the calculation of $Q(\varepsilon)$, but all universal quantities such as dimensions of composite operators.

ACKNOWLEDGMENTS

The authors thank N.V. Antonov for discussions. The work was supported by the Nordic Grant for Network Cooperation with the Baltic Countries and Northwest Russia No. FIN-6/2002 and No. FIN-20/2003. L.Ts.A. and M.V.K. acknowledge the Department of Physical Sciences of the University of Helsinki for kind hospitality.

APPENDIX A: RENORMALIZATION-GROUP ANALYSIS IN THE NORMALIZATION-POINT (NP) SCHEME

1. Renormalization in the NP scheme in space dimension $d>2$

UV renormalization and the subsequent RG analysis may be carried out in different variations. In the body of this paper we have used the MS scheme due to its popularity and because it was used in the (incorrect) nonlocal scheme of Ref. [4]. Consistency conditions like relation (18) ensuring independence of the renormalization constant of the wave number in the MS scheme may appear in a different form in other renormalization schemes.

Here, we shall illustrate this point in the example of the scheme with the "normalization point" (NP) first for the technically simpler case of fixed space dimension $d>2$. In this approach, the renormalization constant Z_ν is calculated from the normalization condition for the 1PI Green function

$$\left.\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}\right|_{p=\mu}=-1 \tag{A1}$$

in contrast to the cancellation of poles in ε in expression (11) in the MS scheme. Then instead of Eqs. (14) and (15), we obtain

$$Z_\nu^{(1)}=u\gamma_{\varphi'\varphi}^{(1)},$$

$$Z_\nu^{(2)}=u^2\gamma_{\varphi'\varphi}^{(2)}-2Z_\nu^{(1)}u\gamma_{\varphi'\varphi}^{(1)}=u^2[\gamma_{\varphi'\varphi}^{(2)}-2(\gamma_{\varphi'\varphi}^{(1)})^2], \tag{A2}$$

and after substitution of Z_ν from Eq. (A2), expression (11) assumes the form

$$\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}=-1+u\,\gamma_{\varphi'\varphi}^{(1)}(s^{2\varepsilon}-1)+u^2[\gamma_{\varphi'\varphi}^{(2)}(s^{4\varepsilon}-1)-2(\gamma_{\varphi'\varphi}^{(1)})^2(s^{2\varepsilon}-1)]+O(u^3). \tag{A3}$$

In the NP scheme, the renormalization constant Z_ν does not depend on $s=\mu/p$ due to the very definition, but cancellation of poles in ε is not obvious in Eq. (11). In the two-loop approximation (A3) with account of expressions (12) and (13), these poles appear in the form $\sim u^2\varepsilon^{-1}\log s$ in several contributions, and the condition of their mutual cancellation is the same relation (18) which ensured the cancellation of the "bad" contributions $\sim u^2\varepsilon^{-1}\log s$ in Z_ν in the MS scheme. As it was previously explained, fulfillment of condition (18) is guaranteed by general theorems of the theory of UV renormalization with local counterterms.

The MS and NP schemes differ by a finite renormalization of the parameters g and ν, therefore all objective physical quantities, in particular critical dimensions (20), calculated in these schemes coincide.

2. Renormalization in the NP scheme for the double (ε,Δ) expansion

Let us start by briefly discussing the possibility to carry out a nonlocal renormalization in the NP scheme. Relations (24) between the renormalized and bare parameters would be preserved in this case, whereas the two independent renormalization constants Z_ν and Z_{D_2} should be determined from the following normalization conditions at $p=\mu$ for the 1PI functions $\Gamma_{\varphi'\varphi}$ (25) and $\Gamma_{\varphi'\varphi'}$ (26):

$$\left.\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}\right|_{p=\mu}=-1,$$

$$\left.\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi'}|_{\omega=0}}{g\nu^3\mu^{2\varepsilon}p^{4-d-2\varepsilon}(d-1)}\right|_{p=\mu}=1. \tag{A4}$$

The problem of the dependence of the renormalization constants on the wave number is absent in such a setup. However, it may be readily checked that conditions (36) and (37) remain necessary to ensure the absence of UV-divergent (at $\varepsilon\to 0$) contributions $\sim u^2\varepsilon^{-1}\log(\mu/p)$ in the renormalized Green functions $\Gamma_{\varphi'\varphi}$ and $\Gamma_{\varphi'\varphi'}$ for arbitrary values of the wave number p.

In the approach with local counterterms, the renormalization constants Z in the NP scheme are determined—instead of the single condition of Eq. (A4)—by the two normalization conditions

$$\left.\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi}|_{\omega=0}}{\nu p^2(d-1)}\right|_{p=\mu}=-1,$$

$$\left.\frac{\mathrm{Tr}\,\Gamma_{\varphi'\varphi'}|_{\omega=0}}{g\nu^3\mu^{2\varepsilon}p^{4-d-2\varepsilon}(d-1)}\right|_{p=\mu}=\frac{u_1}{u_2}+1, \tag{A5}$$

with $\Gamma_{\varphi'\varphi}$ from Eq. (47) and $\Gamma_{\varphi'\varphi'}$ from Eq. (48). From here the renormalization constants Z_ν and Z_{D_2} in the NP scheme follow in the form

$$Z_\nu=1+\left(-\frac{1}{\epsilon}-\frac{3}{2}\zeta+c\right)u_1+\left(\frac{1}{\zeta\epsilon}+c+\frac{3}{2}\right)u_2+\left[\left(-1-\frac{1}{2\zeta}\right)\frac{1}{\epsilon^2}+\left(2c-3\zeta+R-2+\frac{c-1}{\zeta}\right)\frac{1}{\epsilon}\right]u_1^2$$
$$+\left[\frac{2}{(\zeta-1)\epsilon^2}+\left(6+2c-\frac{2}{\zeta}-4\frac{R-2}{\zeta-1}\right)\frac{1}{\epsilon}\right]u_1u_2+\left(-\frac{1}{2\zeta^2\epsilon^2}-\frac{2+R+c}{\zeta\epsilon}\right)u_2^2, \tag{A6}$$

$$Z_{D_2} = 1 + \left[c - \frac{1}{(2+\zeta)\epsilon} - \frac{7}{2} + \frac{5}{(2+\zeta)} \right] \frac{u_1^2}{u_2} + \left(2c - \frac{2}{\epsilon} - 5\zeta - 2 \right) u_1 + \left(c + \frac{3}{2} + \frac{1}{\zeta\epsilon} \right) u_2 + \left[\left(\frac{5}{3+\zeta} - \frac{3}{2+\zeta} \right) \frac{1}{\epsilon^2} \right.$$
$$+ \left(12 - 2c + \frac{-68+2R}{3+\zeta} + 3\frac{c+8}{2+\zeta} \right) \frac{1}{\epsilon} \Bigg] \frac{u_1^3}{u_2} + \left[\left(2c - 1 - \frac{1}{\zeta} - \frac{3}{4+2\zeta} \right) \frac{1}{\epsilon^2} + \left(3R - 2\zeta - 10 + \frac{2c-2}{\zeta} + 3\frac{4+c}{2+\zeta} \right) \frac{1}{\epsilon} \right] u_1^2$$
$$+ \left[\left(\frac{5}{\zeta - 1} - \frac{1}{\zeta} \right) \frac{1}{\epsilon^2} + \left(16 + 4c + \frac{c-4}{\zeta} + \frac{28-6R}{\zeta - 1} \right) \frac{1}{\epsilon} \right] u_1 u_2 + \left(- \frac{1}{2\zeta^2\epsilon^2} - \frac{2+R+c}{\zeta\epsilon} \right) u_2^2, \tag{A7}$$

where $R=-0.168$ and $c=0.2274$ are constants found by numerical integration. These are analogs of expressions (59) and (60) for the renormalization constants obtained previously in the $\overline{MS}$ scheme. It may be readily checked that expressions (A6) and (A7) differ from Eqs. (59) and (60) only by a UV-finite renormalization of the parameters ν, u_1, and u_2.

In the NP scheme, in contrast with the $\overline{MS}$ scheme, the renormalized Green functions have an analytic dependence on the set of parameters ε and Δ, i.e., they do not have factors of the type $a\varepsilon+b\Delta$ in the denominators. This is in accord with the general ideas of the theory of analytic renormalization [14].

In the constants Z of the $\overline{MS}$ scheme with a fixed value of $\zeta \equiv \Delta/\varepsilon = \text{const}$, the dependence on ε is present only in the form of poles $1/\varepsilon$, $1/\varepsilon^2$, etc. Contrary to this, in the constants Z of the NP scheme, regular terms ~ 1, $\varepsilon, \varepsilon^2$, etc. are added to the poles in ε. For calculation of the RG functions and the correction exponents ω in Sec. V on rays (23) with $\zeta=\Delta/\varepsilon=\text{const}$ to the order ε^2, only terms of order $1/\varepsilon$ and 1 are required in the one-loop contributions $\sim u$ to Z, whereas in the two-loop contributions $\sim u^2$ only terms of order $1/\varepsilon^2$ and $1/\varepsilon$ are needed. Expressions (A6) and (A7) are quoted just with this accuracy.

The two-loop expressions (A6) and (A7) for the constants Z with the necessary accuracy together with definitions (66) give rise to the following expressions for the RG functions γ_a:

$$\gamma_\nu = (2 + 3\Delta - c\epsilon)u_1 + (2 + c\Delta + 3\Delta)u_2 - 4(u_1 + u_2)^2(2R + 1), \tag{A8}$$

$$\gamma_{D_2} = \frac{u_1{}^2[2 + (7-c)\Delta + (4-2c)\epsilon]}{u_2} + 2[2 + 5\Delta + (2-c)\epsilon]u_1$$
$$+ [2 + (3+c)\Delta]u_2 - 4\frac{(u_1+u_2)^3(2R+1)}{u_2}, \tag{A9}$$

where the notation is the same as in Eqs. (A6) and (A7). The RG functions (A8) and (A9), contrary to their analogs in the $\overline{MS}$ scheme, do not contain factors like $\zeta+\text{const}$ in denominators, i.e., they are analytic in the pair of parameters ε, Δ, which is a consequence of similar analyticity of the renormalized Green functions. Coordinates of fixed points $u_* \sim g_*$ obtained from Eqs. (A8) and (A9) in the one-loop approximation keep the form of Eq. (77), but the two-loop contributions (which we do not quote) differ from analogous contributions in the $\overline{MS}$ scheme. The eigenvalues $\omega_\pm$ of the matrix ω, however, remain exactly the same as in the $\overline{MS}$ scheme, because these quantities do not depend on the subtraction scheme.

In conclusion, we note that in an attempt to use the NP scheme (A4) in the model [4] with nonlocal renormalization, the inconsistency of this model in terms of the RG functions γ_a would appear in the form of poles $1/\varepsilon$ in the two-loop contributions.

APPENDIX B: Λ RENORMALIZATION AND (ε, Δ) EXPANSION ABOVE TWO DIMENSIONS

As was explained in Sec. IV, in the two-charge model (41) in some graphs the wave-vector integrals diverge at large wave numbers. To regularize such integrals, it is necessary to introduce a cutoff parameter Λ. This may be done, e.g., by restricting Fourier components of the velocity field φ to wave numbers less than Λ in functional (41), which automatically brings about the corresponding sharp wave-vector cutoff in the bare response function (5) and in the bare correlation function (42). It was already explained in Sec. IV that all such Λ divergences are "nearly logarithmic" and appear in the results in the form of powers Λ^α with small (of the order of ε for $\Delta/\varepsilon=\text{const}$) positive exponents α.

The elimination of the Λ divergences may be reduced to a renormalization of the bare parameters. Denoting for brevity the whole set of parameters by e, we introduce the notion of "primary bare parameters" $\tilde{e}_0=\{\tilde{\nu}_0, \tilde{D}_{i0}=\tilde{g}_{i0}\tilde{\nu}_0^3, i=1,2\}$ and "secondary bare parameters" $e_0=\{\nu_0, D_{i0}=g_{i0}\nu_0^3, i=1,2\}$ (see Ref. [16]). The original model is defined by a functional of the type of Eq. (41) with the Λ cutoff introduced and with the "primary bare parameters" $\tilde{e}_0$,

$$S(\Phi) = \varphi'(\tilde{D}_{10}k^{2-2\Delta-2\varepsilon} + \tilde{D}_{20}k^2)\varphi'/2$$
$$+ \varphi'[-\partial_t\varphi + \tilde{\nu}_0\partial^2\varphi - (\varphi\partial)\varphi]. \tag{B1}$$

Renormalization of this model may be carried out in two steps: the first is the Λ renormalization with the aim of removal of all Λ divergences. This amounts to a reorganization of the bare parameters $\tilde{e}_0 \to e_0$, in which the secondary set of parameters is expressed as functions of the primary set,

$e_0=e_0(\tilde{e}_0,\Lambda)$, and vice versa, $\tilde{e}_0=\tilde{e}_0(e_0,\Lambda)$. The correspondence between the two sets $\tilde{e}_0$ and e_0 is bijective perturbatively, therefore any of them may be chosen as the set of independent variables.

If the parameters e_0 are chosen as independent, then in the Green functions Γ of model (B1) expressed in terms of e_0 and Λ, there will be no Λ divergences left (they all will be concentrated in the formulas connecting $\tilde{e}_0$ and e_0) and the limit $\Lambda\to\infty$ may be taken in them with the result of eliminating the cutoff parameter Λ completely from the theory. A trace of the UV divergences which brought about the positive powers of Λ remains, however, in the form of singularities in ε in the Λ-renormalized quantities. This happens because in the Λ renormalization, only terms strictly growing as powers of Λ are removed and collected in the renormalization constants. These terms contain singularities in ε and Δ, although the unrenormalized quantities with fixed Λ were regular functions of ε and Δ. Consequently, in the Λ-renormalized quantities there must be terms left which are singular in ε and Δ, but remain finite in the limit $\Lambda\to\infty$. Thus, the Λ renormalization is a way to trade UV divergences in the form of positive powers of the UV cutoff Λ for poles in ε,Δ and their linear combinations in such a way that in the Λ-renormalized quantities, the limit $\Lambda\to\infty$ may be taken.

The basic conjecture is that the results obtained in this manner for the graphs of the Green functions $\Gamma(e_0,\Lambda=\infty,\ldots)$ (the ellipsis stands for the rest of the arguments, such as frequencies and wave vectors) are exactly the same as those obtained in the "formal scheme," i.e., by analytic continuation of all integrals without Λ divergences on the parameter Δ from the region of small $\Delta<0$ (more accurately $-2\varepsilon<\Delta<0$). In this scheme, the unrenormalized action is functional (41). Such an analytic continuation might be carried out without any reference to the model regularized with the explicit wave-number cutoff Λ, which is common practice in field theories of particle physics. There, however, it is the renormalized parameters which are the physical ones, and their bare counterparts together with the UV cutoff are unphysical auxiliary quantities. In our case, unrenormalized parameters are the physical ones and therefore it is important, in principle, to keep track of their relation to the (auxiliary) renormalized parameters, because the fixed-point values of the latter remain in the asymptotic expressions for various correlation functions and the like.

The next step after the Λ renormalization is the ε renormalization with the goal of removal from all Green functions $\Gamma(e_0,\Lambda=\infty,\ldots)$ poles in ε for $\Delta/\varepsilon=\text{const}$. It is carried out by the transition from the "secondary bare parameters" e_0 (the same notation was used in Sec. IV) to the renormalized parameters $e=\{\nu,g_1,g_2\}$ according to relations (43).

The procedure of the ε renormalization was discussed thoroughly in Sec. IV. Let us now explain in more detail the procedure of the Λ renormalization: the transition from the primary bare parameters $\tilde{e}_0$ to the secondary bare parameters e_0. We emphasize that at this stage, we are interested in the Λ divergences only and regard ε and Δ as fixed parameters without any investigation of singularities in these parameters tending to zero. We shall consider the parameters $\tilde{e}_0$ in the graphs of the functions Γ of model (B1) expressed in terms of e_0 and Λ through the renormalization relations

$$\tilde{D}_{10}=\tilde{g}_{10}\tilde{\nu}_0^3=g_{10}\nu_0^3=D_{10},$$

$$\tilde{D}_{20}=\tilde{g}_{20}\tilde{\nu}_0^3=g_{20}\nu_0^3\tilde{Z}_{D_2}=D_{20}\tilde{Z}_{D_2},$$

$$\tilde{g}_{10}=g_{10}\tilde{Z}_{g_1},\quad \tilde{g}_{20}=g_{20}\tilde{Z}_{g_2},$$

$$\tilde{\nu}_0=\nu_0\tilde{Z}_\nu,\quad \tilde{Z}_{g_1}\tilde{Z}_\nu^3=1,\quad \tilde{Z}_{g_2}\tilde{Z}_\nu^3=\tilde{Z}_{D_2},\tag{B2}$$

similar to Eq. (43). The dimensionless renormalization constants $\tilde{Z}$ in Eq. (B2) are functions of e_0 and Λ expressed in the form of series in $D_{i0}\sim g_{i0}$. The corresponding dimensionless expansion parameters are the following analogs of Eq. (44):

$$\tilde{\alpha}_1\equiv\frac{D_{10}\overline{S}_d}{32\nu_0^3\Lambda^{2\varepsilon}},\quad \tilde{\alpha}_2\equiv\frac{D_{20}\overline{S}_d}{32\nu_0^3\Lambda^{-2\Delta}}.\tag{B3}$$

Therefore, the constants $\tilde{Z}$ in Eq. (B2) assume the form

$$\tilde{Z}_{\nu,D_2}=1+\sum_{n_1\geqslant 0,n_2\geqslant 1}C_{\nu,D_2}^{(n_1,n_2)}\tilde{\alpha}_1^{n_1}\tilde{\alpha}_2^{n_2}\tag{B4}$$

with the dimensionless coefficients $C_{\nu,D_2}^{(n_1,n_2)}$ depending on ε and $\Delta/\varepsilon=\zeta$ only (but in a singular manner). In expansion (B4), not all possible terms are included, but only those which are "Λ divergent," i.e., those with a positive power of Λ in the product $\tilde{\alpha}_1^{n_1}\tilde{\alpha}_2^{n_2}$. From Eq. (B3) it follows that

$$\tilde{\alpha}_1^{n_1}\tilde{\alpha}_2^{n_2}\sim\Lambda^\alpha,\quad \alpha=2(n_2\Delta-n_1\varepsilon),\tag{B5}$$

therefore, for $\varepsilon>0,\Delta>0$, in the Λ-divergent terms with $\alpha>0$ in Eq. (B4) the inequality $n_2\geqslant 1$ holds, i.e., at least one factor with $\tilde{D}_{20}\sim D_{20}$ from Eq. (B1) is present.

From this it follows, in particular, that to the real value $\tilde{D}_{20}=0$ in Eq. (B1) [see the text after Eq. (40)] it corresponds $D_{20}=0$ in Eq. (41), which justifies the derivation of Eq. (82) from Eq. (81) in model (41). We also note that the operation $\tilde{\mathcal{D}}_\mu$ in Eq. (66), defined in Sec. V as $\tilde{\mathcal{D}}_\mu\equiv\mu\partial_\mu$ with fixed parameters e_0, in terms of model (B1) has to be understood as $\mu\partial_\mu$ with fixed $\tilde{e}_0$ and Λ. These definitions are equivalent, because the parameter μ does not enter in renormalization relations (B2).

For the Λ renormalization (B2), analogs of relations (45)–(48) may be written and the corresponding $\tilde{Z}$ calculated at two-loop order. We shall not quote the corresponding results, because explicit expressions connecting the primary ($\tilde{e}_0$) and secondary (e_0) bare parameters are unimportant for the RG analysis of the IR asymptotic behavior in Sec. V, which is carried out in terms of bare parameters e_0 and renormalized parameters e.

In the NP scheme (see Appendix A), the normalization condition (A5) may be imposed in the Λ-renormalized model in the same way as just described for the MS (or $\overline{MS}$) scheme. It is not difficult to see, however, that in the NP scheme the very procedure of the Λ renormalization is actually not necessary. The point is that in this scheme from the

quantity to be renormalized its value at the normalization point is subtracted which automatically leads to a quantity without any UV divergences and thus with a finite—and regular in ε and Δ—limit, when $\Lambda \to \infty$. For renormalized correlation functions, the result is the same as after Λ renormalization, subsequent limit $\Lambda \to \infty$, and final renormalization in the NP scheme. Therefore, the RG functions γ and β are also the same, since their expressions in terms of the renormalized correlation functions coincide in both cases.

[1] L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, and A. N. Vasil'ev, Int. J. Mod. Phys. B **17**, 2137 (2003).
[2] L. Ts. Adzhemyan, N. V. Antonov, M. V. Kompaniets, and A. N. Vasil'ev, Acta Phys. Slov. **52**, 565 (2002).
[3] L. Ts. Adzhemyan, J. Honkonen, M. V. Kompaniets, and A. N. Vasil'ev, Phys. Rev. E **68**, 055302(R) (2003).
[4] D. Ronis, Phys. Rev. A **36**, 3322 (1987).
[5] J. Honkonen and M. Yu. Nalimov, Z. Phys. B: Condens. Matter **99**, 297 (1996).
[6] D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. **36**, 867 (1976); Phys. Rev. A **16**, 732 (1977).
[7] J.-Y. Yuan and D. Ronis, Phys. Rev. A **45**, 5578 (1992).
[8] L. Chen and M. W. Deem, Phys. Rev. E **65**, 011109 (2001).
[9] C. De Dominicis and P. C. Martin, Phys. Rev. A **19**, 419 (1979).
[10] L. Ts. Adzhemyan, A. N. Vasil'ev, and Yu. M. Pis'mak, Teor. Mat. Fiz. **57**, 268 (1983).
[11] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[12] J-D. Fournier and U. Frisch, Phys. Rev. A **17**, 747 (1977).
[13] J. C. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1984).
[14] E. R. Speer, *Generalized Feynman Amplitudes* (Princeton University Press, Princeton, 1969).
[15] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1996).
[16] A. N. Vasil'ev, *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics* (Chapman & Hall/CRC, Boca Raton, FL, 2004).
[17] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasiliev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).
[18] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975), Vol. 2.
[19] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Usp. Fiz. Nauk **166**, 1257 (1996) [Phys. Usp. **39**, 1193 (1996)].
[20] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Zh. Eksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].
[21] V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986); J. Sci. Comput. **1**, 3 (1986).
[22] W. P. Dannevik, V. Yakhot, and S. A. Orszag, Phys. Fluids **30**, 2021 (1987).
[23] Y. Zhou, G. Vahala, and M. Hossain, Phys. Rev. A **40**, 5865 (1989).
[24] D. Carati, Phys. Rev. A **41**, 3129 (1990); Phys. Fluids A **2**, 1854 (1990).
[25] S. H. Lam, Phys. Fluids A **4**, 1007 (1992).
[26] M. J. Giles, Phys. Fluids **6**, 595 (1994).
[27] Y. Nagano and Y. Itazu, Fluid Dyn. Res. **20**, 157 (1997).
[28] G. I. Barenblatt, A. J. Chorin, and V. M. Prostokishin, Physica D **127**, 105 (1999); R. Benzi, S. Ciliberto, C. Baudet, and G. Ruiz-Chavarria, *ibid.* **127**, 111 (1999).
[29] K. R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).